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STORY-WORSHIP PROGRAMS
FOR THE CHURCH SCHOOL YEAR

REV. JAY S. STOWELL, M.A.

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BY

REV. JAY S. STOWELL, M.A.

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STORY WORSHIP PROGRAMS
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TO

BETTY AND STEWART

AND TO THE GIRLS AND BOYS
WHO SHARED IN THE PROGRAMS
AND WHO LISTENED SO ATTENTIVELY
TO THE STORIES HEREIN CONTAINED

FOREWORD

The first three chapters of this book present a discussion of the theory and practice of worship in the church school. The rest of the book contains materials of worship which have been arranged for and used in the church school. They are particularly adapted for one-room schools or schools where two or more departments meet together for the worship period. The stories and talks presented should be of equal value, however, to pastors in the preparation of children's sermons and to others who have occasion to speak to children upon religious themes. They are designed to appeal primarily to Juniors and Intermediates.

It is believed that the careful selection and arrangement of the stories for the accomplishment of very definite ends and the grouping of related material about the important festivals and occasions of the year not only make the book unique, but add greatly to its usefulness and value. The fact that these stories have been successfully used is a testimony to their inherent interest. However, no stories have been included merely because they were interesting. The educational end to be accomplished has in all cases been the ultimate criterion for selection or rejection.

For convenience the arrangement of materials has been made on the basis of calendar months, enough being included for an entire year. Four or five Sundays is not

too long to give to the consideration of a single theme, if a definite impression is to be left upon the lives of the pupils. This plan avoids the scattering of effort through lack of purpose while at the same time the theme changes often enough to avoid monotony. So far as possible stories which embody the idea to be presented have been chosen. The pointing of morals is, therefore, reduced to a minimum.

The author acknowledges with thanks the courteous permission of the editors to use material previously appearing in *The Graded Sunday School Magazine*.

JAY S. STOWELL.

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PART I: THE THEORY AND PRACTICE OF WORSHIP IN THE CHURCH SCHOOL

STORY-WORSHIP PROGRAMS FOR THE CHURCH SCHOOL YEAR

CHAPTER I

WORSHIP IN THE CHURCH SCHOOL

The adoption and use of graded lessons in the church school has created some problems while it has solved others. Superintendents accustomed to uniform lessons often find themselves at sea when graded lessons are introduced. Graded lessons furnish no common theme upon which the thought of the entire school or department may be centered. It is easy to laugh at the superintendent who objects to graded lessons because they furnish him no opportunity to "sum up the lesson," but his objection is based upon something more fundamental than an egoistic desire to exhibit himself. It grows out of the felt need of some unifying thought or activity, something which shall for the moment at least make the group a real unit.

Of course we cannot return to the system of lesson uniformity in order to solve the problem suggested above, but a part at least of the solution is to be found elsewhere, namely in the service of worship. In a strikingly large proportion of our schools there has been no worship. It is true that hymns have been sung and prayers

have been repeated with sincerity and devotion, but it is only recently that any really serious attempt has been made to lead pupils into the experience of worship in the church school and at the same time to train them for the more complete participation in the worship of the church service.

We have to some extent wasted a portion of our already limited time because we had certain habits without an adequate vision of their purpose or their possibilities. Thus we have sung hymns to stir up enthusiasm, to get the pupils into a proper frame of mind to study the lesson, to while away the time until late comers should arrive and for various other reasons. Such purposes are no longer adequate. We are now thinking in terms of worship and the church school is deliberately undertaking the new task of training its pupils to participate intelligently and seriously in common worship.

Thus we find ourselves face to face with two distinct yet related elements in our present situation, namely, a diversified lesson material sometimes perplexing the presiding officer and a fresh vision of the importance of training in worship. Surely this coincidence is significant in that it enables us to perform a hitherto neglected function without the necessity of lengthening our school period and it also provides the unifying element for the departments, which some have felt was endangered by the diversification of lesson material.

Whether the unit for worship shall be the department or two or more departments combined the size and equipment of the school and other local factors will determine. Ordinarily a group larger than a class group is desirable for the most effective common worship. If the department is used as a unit the worship may be carefully

graded. Experience has demonstrated, however, that where there is a shortage of room the Junior and Intermediate Departments may be effectively united for the service of worship and in case of necessity even a more comprehensive group may be handled.

In planning the services of worship for a graded school it is obvious that they cannot be based on the lesson themes. It is probably well that this is so for the arguments against any attempt on the part of the superintendent to duplicate the work of the teacher are more than the arguments for it. We do have a very practical basis, however, for arranging the services, namely, the church and calendar year. The beginning of the new year of work, the Thanksgiving period, Christmas, the great birthdays of February, Lent, Easter, the springtime, summer and many other significant developments make our path here relatively clear. In any plan the determining factor will ultimately be the results to be attained in the lives of the pupils, but these results can best be secured when due regard is paid to the great occasions of the year as they come. At best, the plan will be more or less arbitrary, but a larger load can be carried with the expenditure of less energy if we make use of all the currents which are flowing our way.

In preparing the material of this book the author has worked on the assumption that the attitude which is best described as worshipful grows more naturally out of the consideration of the highest Christian ideals than out of any attempt to picture God to the imagination. One of the important elements in all worship is a sense of fellowship with the Divine. In this connection we may possibly learn something from human fellowships. Two individuals commune best when, forgetting themselves, they

think together about great themes, or work together in worthy tasks. The consideration of a book, the interpretation of a great picture, or the sharing of a deep experience of life will lead two souls into the most intimate relationships. Such communion is at its best when regularity or irregularity of features, style of dress, and all other individual peculiarities are forgotten, and the individuals are conscious only that they are thinking together upon great subjects. Thus it is that the pupil in the church school comes most naturally into communion with God as he tries to think God's great thoughts concerning life, its obligations, and its meaning.

The creation of the attitude of worship is of course the purpose lying deep in the mind of the leader as he plans his service. Curiously enough, however, the purpose of the service may often be best stated in other terms. Real worship, like happiness, friendship and other worth-while things in life, is a by-product. We arrange a service, let us say, with the immediate purpose of making the pupil more self-sacrificing. Through the songs we sing and the prayers we offer and the story which is told we accomplish our immediate aim while at the same time the pupil is led more effectively and more naturally into the experience of worship than would have been the case had there been no immediate purpose to accomplish. The true spirit of worship does not thrive best under the direct attention of the participant. It bears its finest fruitage as a by-product.

In such a service the leader's talk has a distinctive function. It consists in the crystallization of ideals rather than in the impartation of information. It strikes the keynote of the entire service. It must be concrete, adapted to the interests of the pupils and above all deal with

a noble theme. Nothing is better than a story or a biographical incident which relates how Christian ideals have been or are being tested and how they stand the test. Object talks are rarely adapted to the particular needs of this type of service; reviews, lesson summaries, drills and similar matters have their place, but it is not here.

It is essential that the worship period of the school be clearly defined and that other items be kept from intruding upon or interrupting it. Announcements, reports, platform instructions and other related matters may follow or be reserved for the closing session of the school, but they should not be included in the brief worship period. To emphasize this distinctive character of the service, the leader may conduct it from behind the reading-desk, and then step from the platform to an entirely new position on the floor before taking up the matters which follow the service of worship.

The limitations of time and the lack on the part of the pupils of the power of sustained attention make it mandatory that the service be brief. It should begin promptly, move steadily and close abruptly. It should not be allowed to shade off into something quite different. The transition to something else should be definitely marked. Every detail of the service must be planned in advance. The slightest indecision on the part of the leader, a delay of thirty seconds from a cause which ought to have been foreseen and eliminated, or any one of many other circumstances which reveal lack of purpose or preparation is fatal.

Throughout most of the service the pupils will be active participants. Only the few moments during which the leader addresses the school are distinctly his own.

The hymn, the responses, the common prayers, the Psalms repeated in unison, all tend to make the service essentially the pupils' service.

If the services are to move smoothly it will be necessary to plan many of them at least one month in advance of the date on which they are to be used. This gives the classes a chance to learn the Psalms and the other common responses as a part of the regular memory work of the school. A Psalm once learned may be used to advantage many times. There should be enough variety in the service to avoid monotony, but too great variety of order and material will detract from, rather than add to the service. Hymns may well be used several Sundays in succession and Psalms and other parts of the service for a month or more.

To make the worship period rich and at the same time brief enough to avoid intrusion upon the other activities of the school is always a problem. There is a constant temptation to lengthen the service. In the long run, however, the best results would seem to be secured from the short service.

The dignity and beauty of the physical surroundings, the nobility of the songs and prayers, the idealism of the leader's thought, all have their part to play in making it easy for the pupil to worship. The most appropriate room available should be used for the worship period. Very frequently this is the church auditorium. There would seem to be many reasons for using this auditorium for the worship of the church school and no very plausible reason against such use.

A school choir to sing the responses and to lead in the singing will add much to the service. Such a choir may easily be secured in almost any school except the

very small one. In many schools the organization of several choirs will be helpful. Thus a girls' choir, a boys' choir, and a mixed choir may be used in turn.

In conserving the results of the worship period in the school care should be taken that it may be really a training for a more effective participation in the regular church service of worship and not a permanent substitute for it. In some cases an attempt is made to meet this situation by incorporating the worship of the school in the morning church service. This plan has both advantages and disadvantages. Whatever plan is followed, however, it is of the utmost importance that the pupil trained to worship in the church school early find his way into the common service of worship of the church. Any plan which accomplishes this end is worthy of the most careful consideration.

With this precaution we may give our best efforts to the service of worship assured that we are dealing with an experience which is fundamental in the Christian life and that to train in worship is to fit lives for genuine usefulness. Experience seems to indicate that the period of worship properly conducted may become one of the most potent influences for good which is at present to be found in the church school.

CHAPTER II

WORSHIP IN THE ONE-ROOM SCHOOL

Slowly but surely our ideals and methods of church work modify our church architecture. In the meantime, however, our methods of work are directly modified by the sort of building in which we are forced to labor. The number of well-equipped and well-arranged church buildings is steadily increasing but, as yet, many churches are of what may roughly be called the one-room type. A school in such a building may be as carefully graded in its instructional work as a school with a more varied equipment, but it cannot be very thoroughly departmentalized unless various groups meet for the church school work at different hours. This of necessity has a very direct bearing upon any plans for common worship in the church school. All of the members of the one-room school presumably meet at the same time and plans for worship must take into consideration the interests and needs of pupils of all ages. That fact should not, however, lead any worker in a one-room school to abandon all ideal of real worship and to content himself with a mere "opening exercise" in his school. The successful superintendent is not the one who lies down in the face of difficulties, but rather the one who forgets that there are lions in the path and moves straight forward toward his goal. The chances are that he, too, will find that the lions are chained and, therefore, not as dangerous as he had supposed.

It may be granted at the outset that if there are any pupils of kindergarten age in the group they will not be able to enter into the service to any large extent. The presence of very young pupils in the room is usually a problem to be solved and the easiest and probably the best way to solve it is to find some other time or place for their meeting.

To a considerable extent the things which have been said of the beginners are true of the younger primary pupils. They may easily become a disturbing factor in the service and their participation in it is of necessity limited. Experience has demonstrated, however, that primary pupils may be led to participate in many parts of a common service of worship. When this has been accomplished the problem of attention and the consequent disturbance of other worshipers has of course been solved. That the task is an easy one the present writer would not claim, but he does not hesitate to assert that intelligent planning and effort will accomplish wonders even under what seem to be unusual difficulties. There are few experiences more interesting than to watch a group of naturally restive children develop within the space of a few weeks or months habits of attention and joyous participation in worship. The secret lies in a service which is planned carefully, moves steadily, closes abruptly and every element of which is genuinely worth while and to some extent adapted to the comprehension of the group at hand.

This matter of adaptation may seem to be a complicated one when there are so many different ages to consider, but it again is one of the problems which seems more difficult at a distance than when one grapples with it at first hand. The principle is a perfectly simple one, namely, adapt the material not to the primary pupils, nor

again to the adults but rather to the older juniors and the younger intermediates. If such a plan is followed it may be assumed that the juniors and intermediates will benefit most by the service. At the same time both the younger pupils and the older members of the school will find in it so much that meets a common need that their attention will be held and their participation secured. It may be argued that such a service is at best a compromise, and that is true. There are hymns, prayers, responses, stories and exercises which would be most appropriate for use with the younger pupils in the group which can be used in a common service only at the risk of losing a vital grip on the older pupils. Older pupils will listen to such elements once or twice out of curiosity, but they cannot enter into them and very soon the entire effect of the service is endangered, if it is planned to meet the needs of the primary pupils. On the other hand if the service is planned for those above the intermediate age, it is doubtful whether the attention of the younger pupils can be permanently held. There is of course the possibility of planning one Sunday for the primary pupils and the next Sunday for an older group, thus endeavoring sooner or later to have a service which is adapted to every group, but this again is bound ultimately to prove fatal. The success of a service of worship depends so much upon the development of habits of attention, participation and response to certain given situations that a service which is constantly changing its character can never be very fruitful as an agency for training in worship. The successful service in a one-room school is one which is planned and carried out by a leader who keeps consistently before him the interests of the intermediates and juniors in the group. It should not be imagined,

however, that it is all loss and no gain for the school which is forced to hold a common service, especially if the school is a small one. There is an indefinable something which comes from the mingling of various age groups in a common body which to some extent, although perhaps not entirely, balances the advantages of the school which, because of its size and better equipment, can have a thoroughly departmentalized program of worship.

We have taken so much space to indicate the possibility of worship in a one-room school largely because there are so many such schools. We are well aware that in many such schools there is no effective service of worship, but we are equally certain that there might be such a service if as much attention were given to it as its importance would seem to warrant. The supreme difficulty lies not in the inauspicious circumstances but rather in the lack of vision on the part of the superintendent or other person in charge of the common service of the school. If the manager of a factory is devoting all his energy and that of his organization to the production of Springfield rifles, it is needless to hang around the stock room or the shipping room expecting to see a certain percentage of aeroplanes turned out. If you really want to get aeroplanes from the factory you must first convert the manager to your point of view. If he determines to produce aeroplanes he will introduce new machinery and methods or adapt his old machinery to the new end. There is no hope in the situation until the manager really knows what an aeroplane is and determines that he will bend his energies to their manufacture. The analogy holds good in the church school. The reason we do not have worship in church schools is that we have not paused to consider seriously what worship

is, how it might be promoted and what contribution training in worship might make to the lives of our pupils. Our protestations that we cannot have real worship in the average school will have more weight when we have once seriously tried to conduct worship there.

Only recently the writer sat through the "opening exercise" of a large and supposedly well-conducted church school. The session was started somewhat past schedule time, apparently because the leader had numerous matters to attend to just at the last moment. The room was in confusion when he at last took his place on the platform. His first remark was, "Now let's get quiet so that we can begin our Sunday school." The leader of the singing then said, "How many of you boys and girls have song books? Hold them up so that I can see. Well, that's good! Now let us sing number 284—number 284. Has every one found the place? All right now, let us sing." The school then sang the first verse. The leader was not satisfied with the result. He paused before singing the second verse, made some pleasant and witty remarks about the girls outdoing the boys and also about the necessity of smiling instead of looking so solemn. He told a perfectly respectable joke, to which the school superintendent made a witty reply, and the second verse of the hymn was begun. The service was punctuated throughout by sprightly innovations. The pupils were asked to clap their hands during one stanza of a hymn. They responded with enthusiasm and some of the boys even insisted on clapping during a portion of the following stanza, although this did not correspond with instructions received. The prayer was apparently improvised by the leader and while it was a perfectly proper prayer it did not seem to voice the aspirations of boys and girls.

During the prayer one group of intermediate boys, without a teacher, edged toward the door and when they were close enough to it to make their escape assured, they dashed out of the room, leaving the door swinging. At the close of this exercise, which occupied twenty or more minutes of a brief sixty minutes, the classes took up the study of the lesson.

The above case is cited not for the purpose of criticism, for the school in question was above the average in many respects. It was generally well-behaved and well-conducted and the leadership was both intelligent and enthusiastic. The point to be noted is that the leaders were not even attempting to conduct a service of worship, indeed a casual observer would have been perplexed to know just what they were trying to conduct. It is doubtful whether they had any clear idea of the purpose of the "exercise" in which they were engaged. They were evidently carrying out one of those activities which become traditional and which go on and on according to the law of inertia without the necessity of finding a rational ground for their justification.

In contrast to the above was the experiences of visiting a school of practically equal size to the one just mentioned. In this school a definite attempt had been made to develop the idea of worship. The finest room in the building had been chosen for the service. Ushers were at the door and this fact alone gave the pupils a feeling of responsibility and checked before it was born any thought of unseemly exits or meaningless annoyances and interruptions. Late pupils were admitted only at appropriate moments and the mind of the leader was free from all petty matters. At four minutes before the time for opening the service the organist began to play a beau-

tiful and dignified selection. At exactly the appointed hour the school choir of girls sang softly, "The Lord is in his holy temple, let all the earth keep silence before Him." The superintendent rose and said, "The Lord is a great God and a great King above all gods. O come, let us sing unto the Lord. Let us make a joyful noise to the Rock of our salvation. Let us sing together hymn number 64, 'O Worship the King.'" There was no need to ask if every one had a book, for that matter had been attended to by the ushers before the service began, and there was no need to announce the number of the hymn two or three times, for the pupils had been trained to pay attention to the service and one announcement was as effective as three and far more in keeping with the spirit of the service. The entire hymn was sung without interruption; the One Hundredth Psalm was repeated in unison without the aid of books, because the pupils had learned it previously in the classes; another hymn was sung; the pupils were seated, and the leader, in a simple, direct but carefully prepared talk, held up in story form before the vision of the group one of the fundamental attributes of true Christian character. The prayer which followed grew so naturally out of the situation which had been created by the service thus far that it seemed to voice the thoughts and desires of the pupils themselves and as they joined in the Lord's Prayer it was evidently more to them than a mere form. There was an instant of quiet and the school choir sang softly the first verse of "Break Thou the Bread of Life." The moment of hushed silence which followed was an eloquent testimony to the fact that the members of the group had passed through a genuine and uplifting experience in the eighteen minutes since the school had been in session.

The service of worship was over, the leader left the speaker's platform and took his place on the floor. There were announcements to be made and other general matters to be attended to before the lessons of the day were taken up, but they were not sandwiched into elements broken from a service of worship. There will always be many matters of common interest to come before a school which have little or no relation to worship and to try to combine the two is to create a meaningless jumble. Platform instruction and drills are legitimate and useful, new hymns must be learned and memory selections prepared, but these activities can enter into a service of worship only in the most limited way. If the pupil is to enter thoroughly into the spirit of a service of worship he must have it built out of materials which are worshipful, with which he is already familiar and which he can handle with ease. A period may be reserved at the close of the school session for learning new hymns; memory work may be perfected in the class session or in the homes, and hours may be appointed for special training along sundry lines. Thus the necessity for dragging extraneous activities into the service of worship will be obviated.

There is one activity, however, which might well be made a part of the worship period. It is the making of the offering. The difficulty lies in the fact that custom and tradition all suggest a different method of handling the offering in the school. In the Primary and Beginners' Departments it has been made a department function and it has become a real act of worship, as it should be. Among older groups, however, the taking of the offering has developed into a more or less meaningless class function. Our system of reports and often of class rivalries seems to militate against making the offering a gen-

uine act of worship. In many churches the offering has become a real part of the service of worship, a concrete expression of the soul's offering to God. If we were only alert to the lessons which our skilled elementary workers stand ready to teach us concerning the way of handling the offering, and if we had an open mind toward the things already accomplished in the same field in our churches, we might be able to place our Sunday-school offering procedure on a more sensible and pedagogical basis and at the same time further enrich our worship period.

CHAPTER III

WORSHIP IN THE DEPARTMENTALIZED SCHOOL

The problems of worship in a thoroughly departmentalized church school are quite different from those in a one-room school. In general we may say that the problems are simpler in the departmentalized school and that the opportunities for worship and for training in worship are better. This is particularly true in cases where the departments are of sufficient size to lend dignity and spirit to the department activities. The service in an extremely small department is likely to suffer at many points, but particularly in the matter of singing. In some cases it will be better to combine two or more departments for the worship period in order to increase the size of the group even though the equipment would warrant a separate assembly. Provided, however, that the departments are of adequate size, a much more comprehensive and satisfactory program of worship can be put into operation here than in the one-room school. The departments represent more or less homogeneous groups and very definite aims can, therefore, be kept in mind in working out the services.

Before determining what those aims shall be, the entire situation must be carefully reviewed. How much do the pupils already know about worship? Are they accustomed to pray at home? At what age are they expected

to begin to attend the regular church service of worship? Do they actually attend this service? Is there a "children's church"? Do the children come into the regular church service for the period of worship and for a children's sermon and then pass out? All these and many more questions must be answered before we can determine what our aim shall be in our program of worship in the school. Situations are so complex and so individual in character that it is impossible to indicate just what should be the procedure until all the elements are known. There are, however, certain principles which may well guide us along the way.

We may say that the devotional portion of the department program should serve at least three ends. It should at each age meet the pupils' present needs for a common service of worship (unless those needs are already met by some other agency); it should serve as an aid in and an encouragement to the pupils' private devotions; and it should serve as a training for more complete participation in the service of worship of the local church of which the school is a part.

It is evident that we have here a problem which can never be fully solved until a common program for the local parish is worked out after the fullest consultation between pastor, church school workers and parents. Possibly at no point in the entire educational program of the parish is a complete mutual understanding of more importance. It is perfectly easy for well-intentioned parents, teachers and pastors to work at cross purposes here and so to accomplish relatively little. In how many parishes, for example, is there any understanding between parents and teachers as to the proper time for a child to begin to attend church regularly? Sunday-school super-

intendents waste much good energy urging children to "stay to church" when the child knows full well that it is tacitly understood in the home that he is too young to "stay to church" and that he is not expected to stay. If by chance he did stay, however, he would very often find that his coming was a surprise to the pastor and that not an element in the service had been planned with him in mind. A frank conference (or several of them) between parents, church workers and pastors would soon result in a program which would bring our young people into the church service in groups *to stay,* while to-day an unseemly large proportion of them never become regular church attendants at all. Our failure to do anything like team work here would be enormously amusing if it did not border so closely upon criminal negligence.

At just what age pupils should begin to attend the church service we may not all agree. The present writer, however, is convinced that it should be earlier than we have usually supposed. At any rate *a time* should be set for this step and parents and teachers and pastors should work together to make it an event to be looked forward to and experienced with pleasure. We have a definite scheme for promotion from department to department. Why not include in the plan this still more vital step into the regular church service? It is time to discard the much overworked and threadbare argument so often advanced by parents and, sad to relate, sometimes by teachers that "Children must not be forced to go to church or when they grow up they will rebel and never go to church again." Well, we do not need to "force" them then. Instead let us make church attendance so pleasurable and let us train our pupils so well for participation in the service and let us build up in the minds of our

boys and girls and our own minds the assumption that going to church is the expected and the proper thing, that it will be as much taken for granted as the promotion from one department to another. We have talked altogether too much foolishness about Sunday school and church being "too much" for our boys and girls. We do not call five or six hours a day, five days a week, on a hard bench "too much," or, if we do, we still see that our children are on hand for the daily ordeal; we do not call two or three hours in a moving-picture house "too much" for the same young people. No, if we are honest with ourselves, we will be forced to admit that the real reason lies in the fact that we do not consider the matter a vital one and that by our words and actions we have allowed our boys and girls to grow up with the idea that attending church is more or less of a "non-essential industry."

Just how so many parents and teachers got the idea that no pressure should be brought to bear upon children to induce them to attend church is hard to discover, but the fact remains that parents who do not hesitate to bring all sorts of influences to bear upon and influence the conduct of children at every other point, stand altogether neutral when church attendance is mentioned. Only recently the promoter of a junior choir called upon the parents of thirty junior and intermediate boys and girls to consult them in regard to the children's participation in this new venture for the morning church service, a service which incidentally was brief and helpful. In twenty-nine cases the reply was in substance, "I will mention the matter to Johnny, but if he doesn't care to attend church I wouldn't urge him." In one case only did the parent reply, "Johnny will be there." This indifference on the part of parents toward the church service is a

great obstacle to progress just at present, but it is no more than the natural fruitage of a system which has allowed a great gulf to become fixed between the Sunday school and the church and which has made no provision for training boys and girls to participate intelligently in the church service.

This entire discussion of church attendance may seem to be apart from our main topic. As a matter of fact it is most vitally related to it. We cannot go far with our plans for worship until it is settled. We must know whether we are expected at a given age to meet fully the pupils' needs in regard to worship, that is, whether our service in the church school is merely a supplement to or a training for the regular church service. In the opinion of the writer, church-school workers should plan to bring the pupils into the church service at approximately eleven or twelve years of age. Previous to that time the matter of church attendance may be left to the discretion of parents. At that time, or what is even more important, at *some time* definitely determined upon it should be taken for granted that the pupils are to become regular church attendants. The period of transition from the Junior to the Intermediate Department would seem to be the appropriate moment for this step. Surely three or four years later is too late to make the transition an effective one. Of course the event must be planned far in advance of every effort made to prepare the pupils for it. It goes without saying that we must also adapt our church service so that our boys and girls will feel at home in it when they arrive. As one means to this end some pastors are organizing large junior choirs so that the young people have a definite part in the service from the very first. If they have been properly trained for this

transition the service will not suffer, and, even if a real sacrifice were necessary, the matter is one of so much importance as to warrant vigorous treatment. There is no more vital weakness in our entire local program at present than our failure at the point of transition between school and church. Until we grapple with this problem manfully and fearlessly and in a way in which it has not been handled in the past, our efforts will always move in the realm of relative failure or semi-success.

Having thus gotten in mind some of the fundamental elements in our problem, we are ready to consider a little more in detail the developing purposes which should dominate our work at the various steps of the process.

Ideally the child that comes into the Beginners' Department has already had some religious instruction and has begun to develop habits of prayer. Practically, however, the leader must assume that the pupil knows nothing about God and His care and that the child still has his prayer habits to form. In other words, the leader must begin at the very beginning in her efforts to build up in the child's mind something of a conception of a Father God and his care and to develop the desire on the part of the pupil to talk with this God. Active participation on the part of the pupil is bound to be limited here both because of the pupil's lack of experience and his inability to read. Anything of the nature of a "service of worship" will appear at best only in an embryonic stage. The child in the Beginners' Department accepts God very naturally as a member of the group. His thoughts of God are not limited by the philosophical difficulties which often come later in life. Very naturally God enters into every part of the department activities. The church building itself is to him God's house; the offering is in a

very real sense made to God; the songs are sung partly because God likes to hear little children sing, and the prayers are simple talks with God, who is at home, in the church and everywhere and who hears every word that little children say. At no time in the child's entire life is his experience of fellowship with God likely to be more direct and more unrestricted than in these kindergarten years.

The aim of the service at this time should be to meet the child's entire needs for social worship and to stimulate and develop such habits of thought and action as will tend to make prayer natural and easy in the home environment. It is not expected that the child of this age will be found in the church service. The service in his department will be, therefore, in a very real sense his "children's church." The service at this period will meet his present needs, but will only in a general way equip him for further participation in worship. The songs which he sings he will soon outgrow and even the verses and responses which he uses will be forgotten or, if remembered, it will be as the result of accident rather than of design on the part of his teachers. Attitudes and habits will, however, be developed which, if the work of the school is thoroughly coördinated, will be carried over and preserved as he advances to the Primary Department. It is of great importance that the leader of the Beginners' Department understand what is likely to happen in the department to which her pupils are to be promoted, and of equal importance that the superintendent of the Primary Department understand the sort of training which her incoming pupils have had in the Beginners' Department. Only where there is this full and frank understanding between departments can there be such a well-

balanced plan as will conserve the interests of the pupil.

In the Primary Department new factors enter into the situation. The fundamental purposes lying back of the services will remain much the same, but the materials out of which the service is built will be somewhat different. In the Primary Department will begin that process of storing away in the mind some of the materials which will be retained and used through life. To be sure, the chief purpose is to provide a satisfaction for the child's present needs for common worship and to encourage his private devotions. Children's songs which will be discarded later may legitimately be used at this time. This does not mean that we can here use songs of questionable theology, of unwise sentimentalism or of markedly inferior poetic or musical quality. Songs which are simple and childish may be used to some extent, however. If purely kindergarten songs are used, the fact that these songs are selected for the benefit of the younger members of the department should be noted and participation in them usually limited to those members. The third year pupils may be requested to serve as listeners or invited to share in the song which is avowedly used for the benefit of the younger members and in which the older pupils share as an act of helpfulness. Thus the spirit of service and the spirit of worship is stimulated by a single act. Distinctly "children's songs" will here be supplemented by some of those simpler hymns which will remain with the child as a permanent acquisition. Their number will not be large, but those which are used can be written on the blackboard, interpreted by the leader and learned by the pupils. In the same way sentence prayers and responses from the Psalms and elsewhere in the Scriptures may be learned

and used. The Lord's Prayer and one or two classic children's prayers may be learned and used as common prayers. The Twenty-third Psalm and other appropriate selections may likewise be mastered. An appropriate story designed to emphasize some trait of Christian character and fitting into the spirit of the service may be used, but a distinct line should be drawn between this and such departmental instruction as may be given. The service of worship is not primarily a place for instruction. Although some children will at this age attend church with their parents, it may be assumed that most of them will not do so and that the child's entire experience of worship will be in the department of which he is a member.

With the pupil's promotion to the Junior Department several new elements appear. Up to this time we have been largely content to supply the pupil's present needs for common worship and to promote his personal devotional life. We now undertake more directly that training which will fit him to enter into the service of worship in the local church which he will be expected to attend regularly when he graduates from the Junior Department to the Intermediate Department. We must still meet fully his needs for common worship and endeavor to maintain in coöperation with the parents those conditions which will tend to promote the personal prayer life of the pupil, but we must also take up seriously the further problem of preparing the pupil to share in the common service of the church.

At this stage of the process it is important that there be the fullest understanding between the minister of the church, the chorister and the church school workers if there are to be no fatal slips as the transition is made.

Eternal diligence and careful planning are here as elsewhere the price of success. The service of worship in the Junior Department must begin to take on something of the form of the service in the church itself. The pupils must learn and use some of the hymns, the prayers and the responses which he will be expected to know how to use when he enters the church. A conference may reveal the fact that the service in the church itself is in some cases altogether too barren and that readjustments will be needed here as well as in the school. The leader's talk in the department will fill somewhat the same place as does the sermon in the church. In most cases this will be a story or a talk couched in very concrete terms. In some cases it will be an interpretation of the very process of worship itself, for there will need to be much of this if the worship is to be real. Worship is not an experience to be stumbled into through the use of certain forms and methods. It is instead more likely to come as the result of the most detailed planning and the most sympathetic execution of those plans. Very much indeed depends upon the spirit of the leader himself. A leader to whom the service is nothing but a form is not likely to direct any one else along the pathway of true worship. It oftentimes seems to be the finishing touches which really add the most to the effectiveness of the entire service. A brief prayer by the leader growing vitally out of and immediately following the leader's talk, a response sung softly by the choir, or some other detail may add the touch which makes the entire service a reality for the pupil.

At this time perhaps more than at any other conditions are favorable for storing the mind with choice selections for present and future use. Entire hymns may be

learned, common prayers mastered and various Psalms and other selections of Scripture committed to memory. Psychologists are not agreed that the memory is actually any better at this period of the child's development than at other periods, but the way does seem to be a little more clear for memory work now than at any time earlier or later in the educational process. Thoroughness is more important than quantity and the selections learned should be used again and again in the service until they have become a part of the pupil's life.

At this time also it is of the utmost importance that the nurture of the pupil's prayer life be thoroughly correlated with the worship of the school. This is important for several reasons. The worship period itself will not be permanently effective unless the pupil learns by experience the meaning of prayer. At the same time the period of worship can do much to make his habit of private prayer a reality. Unless the habit of private prayer becomes fixed during this period it is not likely to be acquired later. Here again, however, teacher, superintendent and parents must coöperate if the best results are to be secured.

The prayers used in the common service of worship and in the class session should be models worthy of imitation. The purpose and meaning of prayer should be sympathetically explained. The Lord's Prayer and other carefully selected prayers should be studied, learned and used. Topics for prayer should be suggested. After careful training along these lines pupils may be led to write out prayers for class or private use, and under some conditions they may be encouraged to offer extemporaneous prayers in the class room. The wise teacher will know when this can be done and when it cannot be done.

The worship program at this period leads out into so many lines that the responsibilities involved become somewhat embarrassing to one who is not willing to labor and sacrifice that the pupil's religious experience shall be broadened. The significant fact that the child is now being prepared definitely for the service of the church itself must be kept in mind. As a part of the preparation for this advanced step it may be well for the entire department to be taken into the church service occasionally at times specially planned and arranged for in conference with the pastor.

Of course if a regular "junior church" is conducted for boys and girls of junior age the whole program of worship in the school must be modified accordingly. The program of worship can then be abbreviated in the school session and the training suggested above can be carried on instead in connection with the junior church. This leaves more time in the school session for platform instruction, drills and similar matters. There is little to be gained and the possibility of definite loss by conducting two duplicating or uncoördinated services in the school and in the junior church.

When the child graduates from the Junior Department we may assume that if our program up to that point has been successfully carried out he is now a regular church attendant. In the Intermediate Department, therefore, we no longer face the problem of supplying the full needs of the pupil for common worship. We can now shorten the service somewhat if necessary, thus giving more time for the class session, or we may begin to develop the initiative of the pupil himself in carrying on the service in the department. We will still continue to use materials which will be used in the church service, but we may

begin to sacrifice form and finish to some extent for the purpose of developing initiative on the part of the pupils. The reasons for this are sound. We need in our churches many individuals who can conduct public devotional services of various sorts, but more important than this is the opportunity thus provided the pupils to express their religious aspirations in the presence of their fellows and thus insure for those same aspirations a permanence which they would not otherwise have. It is none too early to begin a process of training which will be continued at least through the Senior Department until the service is placed entirely in the hands of the pupils and carried out according to plans formulated by them in close conference with the teachers and department leaders.

The process of development will be very gradual here. Classes or individuals may take charge of and conduct the responsive services. A group may be made responsible for the selection of the hymns for a given Sunday. Recitations or brief talks carefully prepared may be given by individuals. The common prayers may be chosen and led by others, and in due time members of the department may offer their own prayers. The entire order of service may be planned in advance by a given class and ultimately the entire conduct of the service will be placed in the hands of the pupils. It will require much more skill and effort on the part of the department leaders to make such a program effective than to plan the entire work themselves and conduct the service, but church schools are run for the sake of building up efficient Christian characters rather than for the sake of providing easy jobs for workers. A consistent plan for developing the initiative of the members of our Intermediate and Senior Depart-

ments will in six years accomplish much in the building up of strong, self-reliant young Christians. We assume at this time that all the young people in these two departments are attending the regular church service and that it is not necessary to duplicate that particular type of service again in the church school. The habit of private devotions should also have become firmly fixed by this time. We are therefore free to train for leadership as we could not do previously. The service here, while lacking some of the finish of a more formal service, will, in its own way, be just as truly worshipful. It will make a definite contribution both to the *esprit de corps* of the department and to the individual lives of the pupils without attempting to duplicate unnecessarily the form of service held in the church.

There are many questions in regard to worship in the Young People's and Adult Departments which are still unsettled and which can be settled only after all the elements in the local situation are taken into consideration. The existence and program of a young people's society of any sort, apart from the church school proper, the nature and function of the church prayer meeting and many other matters, come up for consideration here. By this time every member of the departments should be tied up definitely to some regular activity of the church. They are no longer in any sense apart from or on probation in the church. It is needless to attempt to duplicate here the functions of the church service. The worship period of the department may become little more than a brief, informal, devotional period with no attempt at a formal service. The distinctive contribution of the church school service of worship has been made during

the earlier years and the time can be devoted to other matters.

It will be seen, therefore, that worship in a church school departmentally organized is a matter of rapidly developing importance up to and through the Junior Department. From that time forward some of its functions begin to be taken over by the church service and the service in the school at first begins to change its character and then diminishes in significance as a formal service. It should be noted of course that worship itself is expected to fill an ever larger and larger place in the individual's life as he matures. The changing emphasis grows out of a coördinated program which hands over to the church service itself certain responsibilities as soon as this can safely be done. Thus the time of the school is freed for other matters of importance and a bridge is created which should make it possible for every pupil in the church school to find his way early into the church service itself. When this is accomplished, one of the most fundamental criticisms of the church school will be rendered invalid.

PART II: MATERIALS AND PROGRAMS OF WORSHIP

CHAPTER IV

IDEALS FOR A NEW YEAR OF WORK

October: First Sunday

ORDER OF SERVICE:

I MUSICAL PRELUDE

II OPENING SENTENCE: (*Sung or recited in unison while seated*)

"The Lord is in his holy temple: let all the earth keep silence before Him: Amen."

III MOMENT OF SILENCE: (*with heads bowed*)

IV LORD'S PRAYER: (*or other unison prayer*)

V THE SHEPHERD'S PSALM: (*Stand and remain standing for the following hymn*)

VI HYMN: "Rejoice, Ye Pure in Heart" (*appropriate for the entire month*)

VII STORY: (*The purpose of this story is to lead the pupils to appreciate the significance of the year of work just beginning and to undertake it seriously.*)

THE GREAT STONE FACE [1]

Little Ernest lived in the eastern highlands. At a distance from his home over on the side of the mountain the rocks took on the form of a great stone face. It was a beautiful, kindly and noble face and the children often

[1] Adapted from *The Great Stone Face* by Hawthorne.

paused in their play to look away to it. Sometimes they spoke of the face in hushed and reverent tones as the "Old Man of the Mountains."

In the evening little Ernest would sit by the fire while his father and mother talked of the great stone face and of the legend which had long been related how that some day a man would appear in the valley who would look just like the great stone face. Then little Ernest would sit quietly and wonder when the man would come who would look like the face, and when he climbed into bed it was often to dream of the face and of the man who was destined to look like it.

In the morning Ernest's first thought was of the face, and at night, after his work was done, he would sit in reverent silence by the side of his cabin, looking away in admiration to the face which radiated so much of benevolence and strength.

Each time that a great man came into the valley Ernest hastened to meet him, lest perchance the man who looked like the face should come and go unrecognized and unhonored. Each time Ernest was forced to turn away in disappointment. None of the great men looked like the face.

The years passed and Ernest was no longer "little Ernest." He became "Neighbor Ernest," the man who was the friend of every one and whom every one trusted. The very spirit of the stone face seemed to be working itself out in Ernest's life. Each year found him a little more thoughtful, a little more kindly and a little more unselfish, although sometimes he was a trifle sad because he had begun to fear that he would never live to see the man who looked like the great stone face.

One day, however, good news came to Ernest. There

was to come into the valley a man who was greater than any man who had ever been there before. "Surely," thought Ernest, "this must be the man who looks like the great stone face," and he hastened to see him. But Ernest was doomed again to disappointment; the great man did not look at all like the face.

Quietly and humbly Ernest started homeward; his last hope was gone. No one could come who was greater than the visitor whom they had entertained that day, and yet he bore no resemblance to the face. It was while Ernest was lost in such thoughts as these that almost instinctively he turned to get another view of the face on the mountains which he had come to love. As he turned, the setting sun fell full on his face and the neighbors, who had known him so long, pausing suddenly, discovered that it was Ernest himself who looked like the great stone face. While he had been waiting and wondering when the man would come who would look like the face, he had himself grown into its likeness.

This little story is more than a story for us; it is a parable of what is continually happening in our own lives. We are continually growing into the actual likeness of the things which we think about, live with and admire. Lives are not made strong and beautiful in an instant. It is a slow and steady process. They are built out of individual actions day after day, and the actions in turn grow out of the kind of thoughts which we think and the kind of friends which we have, until we actually come to look like the things and the people with whom we associate.

It is a sobering thought that our very appearance ultimately depends upon the thoughts which we think and the friends which we cultivate, and yet it is an encourag-

ing thought after all. We come to the church school and we study about Jesus and the things which He said, or we learn about Paul or one of the prophets and we go away and it doesn't seem to have made much difference. But we come again next Sunday, and the next, and the next and so on, and finally we discover, or our friends do, that it has made a difference, that we have been building the very best materials into our lives and the very lines in our faces reveal the fact that we have been living in the presence of the best.

It is something like that that our work ought to mean to us this year. At the end of the year we shall be different individuals than we are now, and what that difference shall be the way that we do our work and the spirit which we put into it in the weeks ahead will largely determine.

VIII HYMN: "The King of Love My Shepherd Is."
IX PRAYER: *By Pastor.*
X RESPONSE: (*Sung by school.*) "Hear us, Heavenly Father, while on Thee we call. May Thy benediction on our spirits fall. Amen."

(*Reports and announcements may be given after this service but not as a part of it.*)

October: Second Sunday

(*The general order of service suggested above may well be used during the entire month.*)

SUGGESTED HYMN: "Lead On, O King Eternal."

STORY: (*Purpose: To hold up at the very beginning of the year an ideal of Christian courage which will inspire the pupils to more vigorous Christian living.*)

IRA STRINGHAM, HERO

It was January 8, 1916, Ira Stringham, a sixteen-year-old boy of Jersey City, was on his way homeward. As he was crossing the bridge over the Morris Canal, he saw two small boys break through the ice below. Quickly he ran down the bank of the canal, threw off his coat and shoes, and started out upon the thin and melting ice. He drew his wallet from his pocket and threw it to the crowd on the bank with the words, "If I don't come back, take this to the police."

Carefully he made his way to the hole in the ice and plunged into the icy water. After a bit he reappeared with a boy in each hand. He pushed them onto the ice, only to have it break with their weight. By this time two telephone linemen had arrived. They threw a rope to Stringham. He tied it about the body of a boy and the boy was drawn to safety. Again the rope went out and the second boy was pulled to the shore. The third time the rope was thrown, but Ira Stringham was too far gone with the cold. He tried to speak, but instead he disappeared beneath the ice.

In the pocketbook thrown to the shore was a card of membership in a Christian Endeavor Society. Thus Stringham's identity was quickly established. It was found that he was a poor boy, that his father was dead, and that he had been obliged to leave school to support a mother and a younger brother and sister.

In spite of these unusual responsibilities, Ira Stringham had not missed a session of his Sunday school in more than six years. He was one of the most earnest workers of his young people's society, and he was active

in two movements for community betterment outside of the church. At his place of employment he was quiet and unassuming, but a task given to him was sure to be done conscientiously and thoroughly.

He died bravely and nobly, but the reason that he died thus was that he had learned to live bravely and nobly. Ira Stringham, a twentieth-century boy, had in him the stuff of which Christian heroes are made. (Adapted from the *Christian Endeavor World.*)

October: Third Sunday

Suggested Hymn: "Fight the Good Fight."

Story: (*Purpose: To hold up at the beginning of the year an ideal of integrity, which will inspire the pupils to renewed zeal for honesty and trustworthiness in daily living.*)

AN HONEST MAN

Nothing is more characteristically Christian than perfect honesty. It is easy to be honest when there is nothing at stake; but there come times when it costs to be honest. Such an occasion came to David Livingstone.

When David Livingstone went to Africa as a missionary he was deeply impressed with the horrors of the slave trade, and he desired to do something to break it up. He had made his way up into the heart of Africa, and he felt that, if he could open a way to the west coast, it would be a step toward the accomplishment of his desire. To take a journey of fifteen hundred miles through the wilderness was not an easy task, and, unaided, it was an impossibility. He must have the help of the natives.

One of the chiefs would provide the men, if he could be assured of their return. Livingstone promised to guide them on the return journey if they would go with him. Accordingly the party was organized and the journey begun. There were jungles to be penetrated, swamps to be traversed, large streams to be crossed, fierce native tribes to be met, and other dangers of the jungles to be encountered. Livingstone was taken down with the fever and his followers were obliged to carry him on their shoulders. After more than six months of exhausting struggle the weary and distressed party reached the ocean. Here at the coast were food, shelter, medical care, and, above all, a boat just ready to sail for England. The captain urged Livingstone to return to England. Livingstone had not heard from his family in nearly two years; he was sick, but he had given his word to his black companions.

He bade farewell to the departing boat, rested for a time, and then began the severe return journey with his friends of the wilderness to whom he had pledged his word of honor which could not be broken.

We are not surprised to know that when Livingstone finally died, his followers carried his body other weary miles to the east coast of Africa, that it might rest in the land from which he had come to them.

Recently a traveler asked the keeper at Westminster Abbey, the place where England's noblemen are buried, "Which grave has had the most visitors during the past year?" "Without question," replied the keeper, "the grave of David Livingstone." And that was the grave of the man who kept his word when it cost something.

October: Fourth Sunday

SUGGESTED HYMN: "O Master, Let Me Walk with Thee."

STORY: *(Purpose: To encourage the habit of kindliness and thoughtfulness.)*

"YOU CALLED ME BROTHER"

Some people are unkind because they are selfish, some because they are ignorant, and others because they are thoughtless and lack imagination. Jesus was always interested in people. Little children, the beggar, the sick man, the woman at the well, all found a friend in Him. It is always refreshing to meet one who has caught this kindly spirit of Jesus.

Recently died a man known as the "Sky Pilot of the Lumber Jacks." Thousands all over this country heard him speak, saw his genial smile, and felt his cordial handclasp. He gave his life to preaching the gospel to the men in the lumber camps and to organizing work for their welfare. Frank Higgins loved men, no matter how rough or uncouth their exterior might be. So big and ruddy looking was Mr. Higgins that few realized how literally he was laying down his life for others. The dread disease which carried him away was working at the very place where the strap of his pack basket, loaded with reading matter for the men, had burned itself into his body.

On what proved to be Higgins' last speaking trip he had become so weakened that it was necessary to call the assistance of a porter.

"I'll have to lean on you, too, brother," said Higgins,

as the colored man took his grip, "for I'm nearly all in," and he placed his arm across the porter's shoulders.

At the train Higgins took out his pocketbook and offered a coin.

"I couldn't take your money, mister," said the porter; "no, sir, I just couldn't."

"Why not?" asked Higgins.

"Why, mister, you called me brother, an' you asked 'bout my wife an' children an' mother. I just couldn't take your money."

It was this kind of love for men because they were men that won Higgins's way to the hearts of those among whom he worked.

One lumber jack whom Higgins had helped to a better life said: "I would lay down my life for Frank Higgins. I love that man." (Adapted from the *Continent.)*

October: Fifth Sunday

SUGGESTED HYMN: "We've a Story to Tell to the Nations."

STORY: *(Purpose: To arouse in the pupils a desire for lives of unselfish service.)*

SERVING TO THE UTTERMOST

Arthur Jackson was one of those big enthusiastic young fellows whom every one is bound to love. There wasn't a lazy bone in his body. At Cambridge University he had been one of the crack oarsmen on the university crew. When his medical course was completed he sailed for northern China as a medical missionary. Here we find him at Mukden in the southern province of Manchuria, January 12, 1911.

Arthur Jackson had only been in China four months, but they had not been idle ones. Measles, mumps, or fever are the same in China as in England or America so that a medical missionary does not have to wait a year or two years as do many other missionaries before they begin their real work. Every day Jackson had been making trips to sick people, performing operations, coaching his Chinese students in football and studying the Chinese language.

On the night of January 12 he was in his room writing to his sister. The bitter cold of the Manchurian winter made the air of his room cold and frosty and occasionally he would rise to walk back and forth or to beat his muscular arms about his great chest. As soon as his hands were warm he went back to his writing. "Whoever invented Chinese," he was saying, "seems to have had an enormous stock of h's, s's, c's, n's, and w's which he no doubt bought at some jumble sale, and it is a wonder that the whole thing has not been sold long ago at another. I can tell you that saying 'Peter Piper,' etc., or any such catch is child's play to managing your s's and w's in Chinese."

For a moment he hesitated, then he plunged again into his letter-writing.

"You may have seen," he wrote, "that the plague is pretty bad in northern Manchuria. We are doing all we can to prevent its coming south. You remember that Mukden is at the junction of the Japanese line running south and the Chinese Imperial Railway running west to Tiensin and Peking. It is an important place as you can see from this sketch." Here he drew a little map.

"Just at this time of the year there are great crowds

of coolies going from their work in the north down into Peking. I am going to examine the passengers to prevent the plague from getting into China. You need not mention this new job I have got to mother, as it would only make her unnecessarily anxious. Of course plague is a nasty thing, but we are hopeful of getting it under now."

Young Jackson rose and paced slowly back and forth in his room. He well knew that in spite of every precaution he might take the dreaded disease, and he knew that, if he did take it, it meant death. A cure had never been known. He looked out of the window on the snowy ground. Away to the west lay the railroad ready to carry its thousands of coolies into China. Who would save Peking and the millions of China? Suddenly Dr. Jackson's shoulders squared themselves. The Master Himself had not saved His own life, why should Arthur Jackson fear to lose his? He walked to the table, sealed his letter and lay down to rest.

The next day his work began. Four hundred coolies were on the first train. Some already had the plague and it was necessary to examine them, separate the infected ones from the rest to die and then take precautionary measure for the others. Dr. Jackson was dressed in white over his fur coat. He wore oilskin boots and gloves, and a shield saturated with disenfectant over his face.

Thus day after day passed for two weeks. On January 23 he lunched with the other missionaries. "Well, we don't make much money out here," he said gayly, "but we do see life." During the twenty-minute lunch he kept them all laughing. He denied that he was tired and then he hastened back to his work. He was in high

spirits. The worst seemed to be over and he had stayed by the job and had made good. He went to bed that night, but the next morning he could not rise. In saving others he had taken the plague. Some hours of terrible suffering passed and Dr. Jackson was taken out and buried under the Manchurian snow.

All over China the news of Dr. Jackson's death was carried. Chinese officials of every rank did honor to the memory of the man who had laid down his life for China. It stirred certain wealthy Chinamen as nothing had ever done before. One man sent $12,000 and later $5,000. Others opened their pocketbooks and a great medical college was established in China and named in honor of the man who counted not his life dear unto himself, but who gave it freely for others, and they a people of a different race and a different language.

CHAPTER V

LEARNING TO BE THANKFUL

November: First Sunday

ORDER OF SERVICE:

I MUSICAL PRELUDE

II OPENING SENTENCES: *(Read by the superintendent or repeated by the entire school)*

"The heavens declare the glory of God, and the firmament showeth his handiwork."

"He causeth the grass to grow for the cattle and herb for the service of man."

"It is a good thing to give thanks unto the Lord."

"Oh, that men would praise the Lord for his goodness and for his wonderful works to the children of men."

III HYMN: "My God, I Thank Thee Who Hast Made" *(school stands and remains standing for the psalm)*

IV THE ONE HUNDREDTH PSALM: *(in unison)*

V UNISON PRAYER

VI STORY: *(The purpose of this story is to encourage the* habit *of gratitude contrasted with occasional thanksgiving.)*

THE MASTER OF THE HARVEST

The Master of the Harvest walked by the side of his fields in the springtime. There had been no rain and the

corn had not come up. A frown was on the face of the Master of the Harvest; grumblings and complaints were on his lips. Surely there would be no harvest.

The little seeds heard the grumblings and said, "How cruel to complain! Are we not ready to do our best when the time comes?"

The wife of the Master of the Harvest spoke cheering words to her husband. Then she went to her Bible and on the fly leaf she wrote a verse.

At last the rain came and the corn sprang up. The Master of the Harvest was satisfied, but he forgot to rejoice and be thankful. His mind was filled with other things.

When the Master's wife asked if the corn was doing well, he answered, "Fairly well," and nothing more. Again the wife opened her book and wrote on the fly leaf.

Very peaceful were the next few weeks. The corn blades shot up, grew tall and strong, and put forth flowers. The ears began to appear.

The Master of the Harvest walked through the fields; he looked at the ears; he saw that they were small, and again he grumbled: "The yield will be less than it ought to be. The harvest will be bad."

The growing plants heard the complaint, and said, "How thankless to complain! Are we not doing our best?" The farmer's wife again spoke cheering words and then went to her Bible and wrote on the fly leaf.

A drought settled over the land and the Master's face grew very dark. He wished for rain. And then the rain came in torrents. Much of the growing corn was forced to bow before the rushing rain, and some of it

could not rise again. The Master of the Harvest railed against the rain. He had not wanted so much.

"Why does he always complain?" moaned the corn plants. "Are we not doing our best?" The Master's wife said nothing, but wrote on the fly leaf of her book.

The weeks passed. The time of harvest came and the barns were filled with golden grain.

One day the Master of the Harvest picked up the book in which his wife had written. He found many verses and among others the following:

"Thou visitest the earth and waterest it; Thou greatly enrichest it."

"Thou crownest the year with thy goodness; and thy paths drop fatness."

"He causeth the grass to grow for the cattle and herb for the service of man."

"It is a good thing to give thanks unto the Lord."

"Oh, that men would praise the Lord for his goodness, and for his wonderful works to the children of men!"

As the Master of the Harvest read, shame filled his soul, and in the place of the old heart of discontent and faultfinding a new heart of thankfulness seemed to grow within him. And the Great Lord of all the Harvests looked down and was glad.

(Adapted from "The Master of the Harvest," by Mrs. Alfred Gatty.)

VII Hymn: "We Plough the Fields and Scatter."

VIII Prayer: *(by pastor)*

IX Response: *(sung by school)* First stanza of "Dear Lord and Father of Mankind" (*or other response*).

November: Second Sunday

SUGGESTED HYMN: "The King of Love My Shepherd Is."

(*The preceding order of service may be used throughout the month.*)

STORY: *(Purpose: To develop the spirit of gratitude by calling attention to our dependence upon the services of others.)*

JAMES AND HIS BREAKFAST

James was not a big boy, and his experience with the world was limited. The day after the cook left for her vacation James came downstairs to find the breakfast steaming hot on the table as usual.

"Did you get breakfast alone?" James asked his mother as they sat down to eat.

"No," said his mother thoughtfully; "I had help; in fact, a great deal of help; more, I suppose, than you would ever imagine."

James didn't know quite what his mother was getting at. At first he thought she was joking, but as he looked at her face he saw that she was serious.

"What do you mean?" he said. "You don't really mean that you had so much help?"

"Yes," replied his mother, "that's just what I mean." Then she began to explain. "Do you see the steaming cup of coffee which your father has? I took the coffee from the can and put the water on it to boil, but that was the easiest part of the whole process. Many months ago, in a country far away, men whom you and I have never seen planted the coffee and tended it through the long months of growth. The coffee berries were

picked and dried, put into sacks and carted away to the train or to the ship docks. Here the coffee was stored away and brought to the United States. Once more it was unloaded, and handled again, and again, and again, as it passed through the hands of wholesalers, roasters, and retailer.

"The grocery boy brought the coffee to the door, but his work, as well as the work of countless others who had handled the coffee sacks with aching backs, would have been of no avail without the help of the coal stokers, the sailors, the captain, the engineers, the train dispatchers, the trainmen, and others. Back of them still were the coal miners, the ship builders, and the railroad builders. If any one of them had failed to do his part I could not have finished the job of preparing the coffee."

By this time James was beginning to understand. When he had heard about the throng of workmen who had been engaged in preparing and bringing the sugar which filled the bowl, when he had thought about the complicated process by which the wheat for the bread, the salt for his egg, and the prepared breakfast food which he liked so well had come to him, he began to be filled with amazement. There still remained the dishes, the knife and fork, the table cloth, the table, the stove, the fuel, and many other things, to be explained. It was clear that more people had been at work at his breakfast than he could enumerate.

Never had he thought of the draymen, the laborers on the street and in the factories, the miners in the mines, and the farmers in the fields as working for him. Now he understood, and his heart was filled with gratitude.

in the world to pay back a part of what others were doing for him.

November: Third Sunday

SUGGESTED HYMN: "We Plough the Fields and Scatter."
STORY: *(Designed to make the pupils more thankful for the common blessings.)*

A LOAF OF BREAD

To-day I want to tell you how one small girl discovered in a very simple way that even the ordinary things which we consume and use from day to day are after all the direct gifts of a wise and benevolent Providence.

A little girl went to her mother one day and said, "Mother, I want to make a loaf of bread all alone."

"All right," said the wise mother. "If you want to make a loaf of bread all alone, go to the kitchen and get permission from the cook and go to work."

The child made her way to the kitchen and there she told the cook, "I want to begin at the very beginning and make a loaf of bread all alone, and I want you to tell me how."

"Very well," replied the cook, "but if you want to begin at the beginning, you will have to go to the grocer and get the flour, as I do."

The little maid went to the grocer and said, "I want some flour, for I am going to begin at the very beginning and make a loaf of bread all alone."

"I can give you the flour," answered the grocer, "but if you want to begin at the very beginning, you had better see the miller, for I get my flour from him."

By this time the little girl was becoming very thoughtful. She hunted up the miller and to him she said, "I started out to begin at the very beginning and make a loaf of bread all alone, but when I went to my mother she sent me to the cook, the cook sent me to the grocer, and now the grocer has sent me to you. I am glad that I have found you because now I can really begin at the beginning for I know you make the flour here from which our fine bread is made."

"Yes," said the miller, "we do make the flour here but we get the wheat from the farmer and if you really want to begin at the beginning you will have to go and see him."

The little girl was determined that she would not give up her task even though it was proving a longer and harder one than she had anticipated, so she made her way to the farmer, this time confident that she had reached the "beginning" of her loaf of bread.

The farmer was very kind and courteous but he shook his head in a discouraging way. "Yes," said he, "I plant the seed and tend it and I gather the harvest but the sunshine and the showers and the little germ of life which makes the seed grow all come from God, so I am afraid, my little lady, that you will never be able to begin at the beginning and make a loaf of bread or in fact anything else. Back of our food, our clothing, the lumber in our houses, the coal in our furnaces and all the other things which we use and enjoy is God. We cannot do even the simplest thing such as to make a loaf of bread without his help."

The little girl had not yet got started on her loaf of bread, but she had done an even more important thing; she had discovered a great truth.

November: Fourth Sunday

SUGGESTED HYMN: "Praise the Lord; Ye Heavens Adore Him."

STORY: *(True thanksgiving grows out of contentment. The purpose of this story is to make the pupils contented with the circumstances in which God had placed them.)*

HOFUS, THE STONE-CUTTER

Hofus was a poor stone-cutter in Japan. His food was coarse, and his clothing was plain, but he was happy and content with his lot, until one day he took a load of stone to the house of a rich man. When Hofus saw the evidences of wealth, he cried, "Oh, that Hofus were rich!"

As Hofus said this a fairy cried, "Have thy wish!" and immediately Hofus was rich. He ceased to work and lived in luxury and contentment, until one day he saw a prince with a snow-white carriage, snow-white horses, a golden umbrella, and many, many servants.

Then cried Hofus, "Oh, that Hofus were a prince!" No sooner had Hofus uttered his wish than he became a prince. Hofus was happy and content as a prince, until one day, riding in his beautiful carriage under his golden umbrella, he sweltered and burned in the rays of the sun.

"The sun is greater than I," cried Hofus. "Oh, that Hofus were the sun!" Immediately Hofus became the sun, and he was happy and content, until a great cloud came and entirely hid the sun.

Then cried Hofus, "The cloud is greater than I. Oh, that Hofus were the cloud!" Immediately Hofus be-

came a cloud. Hofus was happy and content as a cloud, until the cloud fell as rain and swept everything before it except a great rock which stood unmoved by the torrent.

Then cried Hofus, "The rock is greater than I. Oh, that Hofus were only a rock!" Immediately Hofus became a rock, and he was happy and content as a rock, until one day a stone-cutter came to the rock and began to split it.

Then cried Hofus, "The stone-cutter is greater than I. Oh, that Hofus were a stone-cutter!" Immediately Hofus became a stone-cutter, as he had been before, and this time Hofus was really happy and content, for he had learned that there are disadvantages in every station in life, and that the best place for each of us is exactly where God has put us.

(Adapted from a Japanese legend.)

CHAPTER VI

THE GIVING LIFE

December: First Sunday

ORDER OF SERVICE:

I MUSICAL PRELUDE
II HYMN: "Hail to the Brightness of Zion's Glad Morning."
III THE BEATITUDES *(Repeated in unison)*
IV PRAYER
V HYMN: "Christ for the World We Sing"
VI STORY: *(The purpose of this story is to encourage the "giving" instead of the "getting" ideal of life.)*

THE MEANING OF SUCCESS

Some years ago there died in the city of New York one of our earliest and most famous millionaires. He had begun his life as a poor boy and had become very wealthy. While he lived he was held up as an example of success, and schoolboys were stimulated to hope big things because of his achievements. When he died a leading New York newspaper said in an editorial: "Not a single human interest has suffered in the least by the death of Mr. ——."

This man had made his money by artificially depressing the stock of certain great properties and then buying

while cheap and holding or selling at a profit. His work had not helped to build up industry or to meet the needs of men. His profits had always meant the loss of some one else. His money he had kept for himself and his family. When he died a daily newspaper awoke to the fact that the life of this man who had been rated a success had proved a miserable failure.

Across the Atlantic ocean there lived another man. He never succeeded in accumulating money or other possessions. In fact, he could hardly get enough for the necessities of life. Many times he was in the most distressing situations for lack of money. His clothing was threadbare; his food was inadequate. His life had seemed to be one long series of failures. He saw his plans collapse one after another; his friends spoke of him in pitying tones; other people rated him as a failure. Finally he died a broken-down and discouraged old man.

Not long after his death, however, his grateful countrymen erected a monument in his honor, and they put upon it words something like these: "Savior of the poor, father of the fatherless, educator of humanity; man, Christian, citizen; all for others, nothing for himself; peace to his ashes; to our beloved Father Pestalozzi." This man who had never been able to accumulate much of this world's goods had succeeded because he had been able to give much to the world.

One man failed, not because he was rich, but because he led the selfish "getting" life. The other man succeeded, not because he was poor, but because he led the unselfish "giving" life. Each demonstrated in his own way the truth of Jesus's words when he said, "It is more blessed to give than to receive."

VII PRAYER: *(by pastor or superintendent)*
VIII RESPONSE: *(by the school)*
"Hear us, Heavenly Father" *(or other response).*
IX HYMN: "It Came Upon the Midnight Clear."

December: Second Sunday

SUGGESTED HYMN: "Joy to the World."
STORY: *(Purpose: To encourage the habit of kindliness.)*

A FOLLOWER OF JESUS

A train was pulling into the depot. On the platform stood a very small, crippled fruit boy. His basket was filled with fruit and nuts ready to sell to the passengers. The train had not yet come to a full stop when a business man had swung himself from the train and in his haste collided with the boy on the platform. The basket was overturned and its contents scattered.

The man saw what had happened, but, as a crippled fruit boy was the only one concerned and as the man was in a hurry, he walked away toward the city without a word.

Just then the train stopped and a traveling man alighted. He, too, had important business in the city, but here was a boy in trouble. The traveling man comprehended the situation in a glance—the scattered fruit, the crippled boy, the distress on his face, and the tears in his eyes.

The man said nothing, but he set down his bag, and quietly, but rapidly, he assisted the boy to gather and replace in the basket the fruit and packages which could be rescued from amid the hurrying feet. The task was

completed and the traveler was about to leave when he reached into his pocket and, taking out a silver dollar, he placed it on top of the basket.

As he did so the boy looked up through his tears into the face of the man and said, "Say, mister, be you Jesus?"

"No," said the man, "I am not Jesus, but I am one of his followers, and, as I go about, I try to do the things which I think he would do if he were here."

December: Third Sunday

SUGGESTED HYMN: "Silent Night! Holy Night!"

STORY: *(Purpose: To lead the pupils to think of kindliness to others as a service to Jesus.)*

RACHEL AND DAVID

The following is an adaptation of an old legend, but, like all worth-while legends, it contains more of truth than of fiction:

In a place far away there lived a long time ago two little children, Rachel and David. They lived in the country and they were very poor. Their food was coarse, and their clothing was worn; but they were happy and content, for their home was a home where love dwelt.

One cold winter evening as the family was gathered about the rough board table for the evening meal of coarse bread a sound was heard without. Rachel and David listened and then hastened to the door and peered out into the darkness. There in the snow was a little child. "I am hungry and very cold," said the child.

Quickly they brought the little child within. They

placed him by the fire to warm himself and then they shared with him their scanty evening loaf.

Should they keep him all night? There was no extra bed.

"Yes," said the children, "he can have our bed." And so they tucked the little stranger away in bed and then they lay down on the hearth to sleep.

They did not know how long they had slept when they were awakened by the most wonderful music which they had ever heard. It seemed to be without and within and everywhere. They sat up and listened. Then they looked toward the bed where the little child was sleeping. They were amazed to see that his garments were of spotless white and that about his head was a circle of wonderful light.

Then they knew that in taking the little stranger into their home, in sharing with him their fire, their food, and their bed, they had really been ministering unto the Christ Child.

Then, also, Rachel and David remembered the words which they had read in the Book: "Inasmuch as ye have done it unto one of the least of these my brethren, ye have done it unto me."

December: Fourth Sunday

SUGGESTED HYMN: "Hark! The Herald Angels Sing."
STORY: *(Purpose: To deepen the appreciation of this wonderful story, "which never grows old.")*

THE CHRIST-CHILD

Now it came to pass in those days, there went out a decree from Cæsar Augustus, that all the world should be enrolled. . . . And all went to enroll themselves,

every one to his own city. And Joseph also went up from Galilee, out of the city of Nazareth, into Judea, to the city of David, which is called Bethlehem, because he was of the house and family of David; to enroll himself with Mary, who was betrothed to him. . . . And it came to pass, while they were there, the days were fulfilled that she should be delivered. And she brought forth her first-born son; and she wrapped him in swaddling clothes and laid him in a manger, because there was no room for them in the inn.

And there were shepherds in the same country abiding in the field, and keeping watch by night over their flock. And an angel of the Lord stood by them, and the glory of the Lord shone round about them; and they were sore afraid. And the angel said unto them, "Be not afraid, for behold, I bring you good tidings of great joy which shall be to all the people; for there is born to you this day in the city of David a Savior, who is Christ the Lord. And this is the sign unto you: Ye shall find a babe wrapped in swaddling clothes, and lying in a manger." And suddenly there was with the angel a multitude of the heavenly host praising God, and saying,

"Glory to God in the highest,
And on earth peace among men in whom
he is well pleased."

And it came to pass, when the angels went away from them into heaven, the shepherds said one to another, "Let us now go even unto Bethlehem, and see this thing that is come to pass, which the Lord hath made known unto us." And they came with haste, and found both Mary and Joseph, and the babe lying in the manger. And when

they saw it, they made known concerning the saying which was spoken to them about this child. And all that heard it wondered at the things which were spoken unto them by the shepherds. But Mary kept all these sayings, pondering them in her heart. And the shepherds returned, glorifying and praising God for all the things that they had heard and seen, even as it was spoken unto them.

December: Fifth Sunday

SUGGESTED HYMN: "Our God, Our Help."

STORY: *(Purpose: To arouse the desire to make the new year better than anything which has preceded it.)*

THE NEW YEAR

The passing of the old year and the coming of the new is always an interesting occasion. There is something almost mysterious about the midnight moment which seemes to separate but which really connects the one with the other.

It is wonderful when we pause to think of it how life is measured out to us in terms of new days, new weeks, new months and new years. Each one seems to be a sort of challenge to us to forget the mistakes of the past and to move forward to better things.

If the school record of the past year has been lower than it should have been, the new year offers a chance for us to redeem ourselves.

If our work for our employers has been done indifferently and carelessly, the new year holds out the opportunity for conscientious and faithful service.

If the home life has been soured by our selfishness and

thoughtlessness, the new year bids us make it sweet and clean.

The new year is full of allurements to better things along many lines.

Just what the year, which is about to begin, may have in store for us we do not know; God does not permit us to see the end from the beginning.

Our duty is not to worry about the future, but to do the thing which lies just before us and do it so well that we shall be ready with a clear conscience for whatever may follow.

We stand at the beginning of the year in somewhat the same position as did Christian in Bunyan's *Pilgrim's Progress.* He was anxious to know the way, so he consulted Evangelist.

"Do you see yonder wicket gate?" said Evangelist.

Christian strained his eyes to look, but was forced to reply, "No, I see nothing."

"Do you see yonder shining light?" said Evangelist.

Again Christian gazed intently forward, "Yes, I think I do."

"Then," said Evangelist, "keep the light in your eye and go up directly thereto; so shalt thou see the gate."

That is a picture of life. We do not have to do the duties of next October now, nor even the duties of February. The only task which faces us is the one immediately before us, and there is always a shining light, even though it be faint, to make that duty clear. As we follow it doing the duties one by one as they come we shall find many wicket gates opening before us.

During the months ahead some of you will be passing from one grade to another, some from grammar school to high school, some from high school to college,

and others out into the world of business and labor.

Always there is some new task, some new duty, some new opportunity. The only people who fail are those who through discouragement or too great satisfaction over past achievements cease to strive and lie down to rest when they ought to be straining every fiber to advance.

There is an old legend of a shepherd who one day, working in the field, caught a glimpse through the clouds of the place where at the top of the mountain the gods lived.

"Oh," said the shepherd, "would that I could dwell in the heights with the gods!"

Suiting his action to his wish he left his humble abode and set out for the mountain top. The way was steep and long and many thorns were in the path, but the shepherd overcame all difficulties. At last he stood on the heights and sure enough the gods were there.

The gods commended the shepherd for his efforts, and the shepherd, well pleased with himself and with his achievements, lay down to rest. How long he slept he did not know, but when he wakened he was enveloped in a cold mist. The gods were nowhere to be seen. The shepherd called in vain, but finally for just an instant he caught a glimpse through the mist of the gods on a still loftier peak far away.

The shepherd cried out in grief to think that in spite of his hard climb he was still far from his goal. Then he heard a voice from the cloud say: "Foolish mortal, dost thou not know that he who would dwell in the heights with the gods must not sleep but must forever climb higher and higher?"

This is the challenge of the new year to us!

CHAPTER VII

DOING ONE'S DUTY

January: First Sunday

ORDER OF SERVICE:

I MUSICAL PRELUDE

II OPENING SENTENCES: "For the Lord is a great God, and a great King above all gods. O come, let us sing unto the Lord; let us make a joyful noise to the rock of our salvation."

III HYMN: "O Worship the King."

IV THE ONE HUNDRED TWENTY-FIRST PSALM: *(repeated in unison)*

V HYMN: "O Master, Let Me Walk with Thee."

VI STORY

SERVICE THROUGH THE DAILY TASK

Nothing will more surely make for success during the coming year than a sense of the importance of the task at which God has placed us. A task becomes of supreme value in itself and a stepping stone to something more worthy only as we put into its performance our best of skill and devotion.

In one of our western cities there sat one night, several years ago, a number of telephone girls. As they responded to the calls during the busy evening hours or talked together during the quieter hours it was not apparent that they were very different from each other.

In an instant everything was changed. In passing a window an operator discovered that a fire had broken out in the business section next to the telephone building.

"The whole town's on fire, girls! Run for your lives!" shouted the operator. And the girls did run; deserting their posts, they started pell-mell down the stairs—all except Rose Coppinger.

A moment before Rose had been one of a group. Now she was in a class by herself. Calmly she sat at her post, called up the fire department, and then began summoning aid from the country about.

No one thought of Rose Coppinger until the telephone building was in flames; then some one remembered. Desperate efforts were made, and the operating room was reached. Rose was found unconscious at her post, with the telephone receiver strapped to her ear. Wet blankets and restorative measures saved her life and brought her back to consciousness.

When the fire had finally been brought under control, with a part of the town still unburned, the heroism of the telephone operator began to stand out. A committee of citizens presented the brave girl a generous purse "for heroic services in saving the town from absolute destruction."

Better than any purse to Rose, however, was the consciousness that she had taken her humble task seriously, and that when the test came she had been faithful to duty even at the risk of her own life.

(Adapted from the "Christian Endeavor World.")

VII PRAYER: *(by pastor or superintendent)*
VIII RESPONSE: *(by the school or school choir)*

January: Second Sunday

SUGGESTED HYMN: "Take My Life, and Let It Be."

FAITHFUL UNTO DEATH

"Policeman Joseph Mangan, a young man not long on the force, is dying in the hospital at Seventh Avenue and Sixth Street as the result of injuries received while rescuing two small children from the third floor of a burning building early yesterday morning." Thus read the newspaper paragraph. That paragraph did not tell the whole story, however.

Young Mangan was ambitious. He wanted to make a place for himself in the world. He passed the examinations, and at the age of twenty-six was a member of the police force of our greatest city. His record for the short time he was in the service was most excellent. The prospect of a long future of usefulness with a pension upon retirement was before him.

Soon after taking up his work he was serving on night duty. At 2:30 A. M., as he covered his territory, he discovered a building on fire. He turned in an alarm and then hastened to arouse the occupants. The second floor was soon cleared, and Mangan rushed to the third. The fire now was burning fiercely, and Mangan and another policeman who had now arrived were nearly exhausted when the last family was guided to safety down the fire-escape.

Just then a mother began to scream: "My babies! My babies!"

"Why didn't you tell me?" cried Mangan, and darted into the building.

He found little Edward crying in the middle of the bedroom, and rushed with him to the street. Then he hurried back through the smoke for Dorothy. He found her in her crib nearly suffocated.

By this time Mangan himself was almost overcome. He reached the stairway, and there he staggered and fell. Mangan's skull was crushed, but little Dorothy, landing on top of him, was uninjured.

The newspaper paragraph tells the rest of the story.

That sort of devotion to duty we have come to expect on the part of policemen, firemen, and missionaries. Because our work is of a different character we sometimes excuse ourselves with a lower standard. Anything short of service to the uttermost, however, is less than our best. Jesus counted not his life dear unto himself, and, however humble our task may be, we may put into it the same spirit which Jesus put into his life and which Joseph Mangan put into his task.

January: Third Sunday

SUGGESTED HYMN: "The Son of God Goes Forth to War."

STORY:

THE JOY OF DUTY WELL DONE

One bitter cold winter morning the wife of a lighthouse keeper was watching the light while her husband caught an hour of sleep. In the gray dawn she saw a small schooner on the rocks. It had been wrecked by the angry sea. Three sailors were clinging to the rigging. Apparently the rest of the crew had been drowned.

Should the woman waken her husband? If she did she felt sure that he would take his lifeboat and attempt

the rescue of the men on the wreck. She doubted if a boat could live in such a sea. Her husband might be drowned. If she waited a little the men would probably succumb to the cold and slip off into the sea, or the wreck itself might be sent to the bottom by some huge wave. She hesitated between her love for her husband and her sense of duty.

The stern habit of fidelity to duty through long years of vigilant service overrode every other consideration. She wakened her husband. He went at once in the life-boat, and finally succeeded in bringing all three of the unfortunate victims of the wreck off alive.

The comfort and pleasure which the memory of that awful morning brought to the husband and wife, it is said, was more than that which any five years of softer delights had ever brought. They had learned the joy of meeting unflinchingly a duty which was hard and relentless, and with such a joy the world has few that can compare.

(Adapted from an article by Charles R. Brown in the *Congregationalist.*)

January: Fourth Sunday

SUGGESTED HYMN: "Fight the Good Fight."
STORY:

THE CALL OF DUTY

It would be a mistake to imagine that all of the fine and noble things in life are done in the lime-light. There are thousands of instances of heroic devotion to duty which never come to public notice or receive attention merely by accident. Fortunately, duty well done carries its own reward, even when the devotion is unrecognized

by those who are closest at hand. To be loyal for the sake of winning the approval of others is to be disloyal to our best.

Nearly a generation ago in one of our eastern colleges a very promising young man had reached his junior year. He was one of the most brilliant men in the class, and he was a general favorite with his companions. His prospects for the future were bright.

In a twinkling everything was changed. Word came that his father had been stricken with paralysis. There was no one to care for the father and mother, and no one to work the little New England farm except this educated, trained young man.

Putting aside ambition, prospects, chosen profession, and all of the things toward which he had been working, the young man took up his new tasks. He became a farmer and cared tenderly for his father and mother as long as they lived. When they passed away the years had already set their seal upon the fate of this man, now no longer young.

On the same farm to which he was called so long ago the man, now well along in years, still lives. His life has not been all that at one time he expected it would be. There has not been much of romance or glamour about it. He is a lonely old man, but he has experienced the satisfaction of knowing that when the call of duty came he was true even though the cost was great.

It is because there is so much of this sort of Christian heroism in the world that we find it so good a place in which to live.

(Adapted from an article by Howard B. Grose in the *Congregationalist.)*

CHAPTER VIII

CHRISTIAN PATRIOTISM

February: First Sunday

During the month of February two occasions of nation-wide significance occur. These are the birthdays of Lincoln and Washington. The month furnishes, therefore, a special opportunity for building up an ideal of manhood and Christian patriotism. To this end we have chosen for consideration in addition to Lincoln and Washington, two famous Americans who, each in his own way, helped to advance the kingdom of God in our land.

ORDER OF SERVICE:

I MUSICAL PRELUDE

II OPENING SENTENCES: *(Sung or repeated in unison)*

"The Lord is in his holy temple. Let all the earth keep silence before him."

III MOMENT OF SILENCE *(with bowed heads)*

IV THE LORD'S PRAYER

V HYMN: "My God, I Thank Thee."

VI STORY:

BOOKER T. WASHINGTON

The ability and the opportunity to overcome seemingly insuperable obstacles and to achieve great things from

humble beginnings have always been characteristic of American life. Few Americans have achieved more than did Booker T. Washington. Born a slave, this man made his way against personal opposition and difficulties until his personality and his achievements were recognized both in the United States and abroad.

As a boy Booker T. Washington did not have many advantages. In fact, he did not even have a name, but chose one for himself. He got his first lesson in numbers from learning the number "18" on the end of a salt barrel. His desire for learning, however, outran his opportunity, and it was a happy day for him when he heard of Hampton Institute and resolved to go there. Walking, riding, working his way, using a hole under a sidewalk for a lodging place, he at last reached the place of his dreams, only to be confronted with the possibility of rejection. His entrance examination was the sweeping and dusting of a room. He used to say that he swept the room three times and dusted it four times. When his examiner returned to go over the room with her pocket handkerchief, not a particle of dust could be found.

When Washington had completed his course at Hampton he decided to give himself to work for his own race. The method of that service was still undecided. After deliberation he chose not the easy way, but the way what seemed to be right. He was sure that the Negro must be taught to work efficiently with his hands, but that was just the thing the Negro did not care to learn. The Negro wanted an education, not that he might work with his hands, but that he might escape that sort of work. Washington chose the unpopular course because he felt that it was the right course. He determined to

establish an industrial training school for the Negro. The beginnings of this school were almost too humble for belief, but the faith of Booker Washington was of the sort which removes mountains and which builds schools out of little or nothing. Slowly, and through much labor and toil, he saw his institution grow into one of the great institutions of our land.

In the list of our great Americans the name of Booker T. Washington will always appear as one whom difficulties could not daunt and who, with opportunities which seemed meager indeed, became the leader, champion and prophet of more than ten million people.

VII HYMN: "O Beautiful for Spacious Skies."
VIII PRAYER: *(by superintendent or pastor.)*

February: Second Sunday

SUGGESTED HYMN: "Our God, Our Help in Ages Past."
STORY:

ABRAHAM LINCOLN

Abraham Lincoln was one of those choice spirits who belong to all ages. The stories connected with his life are many and fascinating and of the sort which never grow old. Nowhere, however, do we see the real Lincoln better than in his trust in and dependence upon God.

During the struggle preceding his election Mr. Lincoln went frequently to his friend, Newton Bateman. On one of these occasions he was much depressed by the fact that many of the best people seemed to be opposed to his election. His feelings were greatly stirred. He walked the floor for some minutes, and then, with a trembling voice and with cheeks wet with tears, he said:

"I know that there is a God and that he hates injustice and slavery. I see the storm coming, and I know that his hand is in it. If he has a place for me—and I think he has—I believe I'm ready. I am nothing, but truth is everything. I know that I am right, because I know that liberty is right, for Christ teaches it. Douglas doesn't care whether slavery is voted up or down, but God cares, and humanity cares, and I care; and with God's help I shall not fail."

It was this consciousness that he was working with God and that God was working with him which made Lincoln such a tower of strength in the face of circumstances of almost inconceivable difficulty. Much has been written about Lincoln, the rail splitter, the farm hand, the store clerk, the postmaster, the stump speaker, the story teller and the statesmen. We know of his humor, his wisdom and his unfailing honesty. It will do us good occasionally to get back of all of these and to think of Lincoln as the reverent man who believed so firmly and depended so constantly upon God.

It was upon such a faith that his reverence for the common people and for the laws of both God and man was founded. With such a background it was natural for him to say:

"Let reverence for the laws be breathed by every American mother to the lisping babe that prattles on her lap; let it be taught in schools, in seminaries and in colleges; let it be written in primers, spelling books and in almanacs; let it be preached from the pulpit, proclaimed in legislative halls and enforced in courts of justice. And, in short, let it become the political religion of the nation; and let the old and the young, the rich and the poor, the grave and the gay of all sexes and

tongues and colors and conditions sacrifice unceasingly upon its altars."

Upon the death of Lincoln, Henry Ward Beecher said:

"Again a great leader of the people has passed through toil, sorrow, battle and war, and has come near to the promised land of peace into which he might not pass over. Who shall recount our martyr's sufferings for this people? By day and by night he trod a way of danger and of darkness. On his shoulders rested a government which was dearer to him than his very life. Upon thousands of hearts great sorrows and anxieties have rested, but not on one such and in such measure as on that simple, truthful, noble soul, our sainted and beloved Lincoln.

"Never impassioned, nor yet despondent, he held on through four dreadful, purgatorial years wherein God was cleansing the sin of his people as by fire."

Such was the man whose memory we honor at all times, but particularly on his birthday.

February: Third Sunday

SUGGESTED HYMN: "America."
STORY:

GEORGE WASHINGTON

In the year 1745 a thirteen-year-old boy in the new country of America wrote for his own personal guidance the following: "To labor to keep alive in my breast that little spark of celestial fire called conscience." Few individuals ever succeeded better in carrying out such a purpose than did this boy, George Washington. In personal relations, in business, in public life, Washington's fidelity to the inner voice was his outstanding character-

istic. The business transactions conducted by Washington were many and of great variety, yet it is said that there is not a case on record "of any attempt on his part to get the better of any of his fellows." The product of his estate were passed without inspection, so well known was his fidelity to truth and honesty. The inner voice which Washington followed was more to him than a mere mechanism for warning him of the wrong path. God was real and personal to him. In all of the crises of life, Washington's faith in a God of love stood by him. Without such faith he could not have borne the burdens which were his.

During the terrible winter at Valley Forge, Mr. Isaac Potts was one day walking over his estate when, in the woods by the side of a stream he heard a very solemn voice. He approached the spot, and there discovered Washington's horse tied to a tree. At a little distance in a thicket he saw Washington on his knees in the snow in prayer. Reverently Mr. Potts turned away, and as he returned to the house he burst into tears, saying to his wife, "If there is any one on this earth that the Lord will listen to, it is George Washington."

This fine faith of Washington's was the direct result of the training which he had received from his mother. The story of his relations with his mother is a beautiful one. One incident is typical.

One day in the year 1789 word arrived at Mount Vernon that Washington had been unanimously elected the first President of the United States, and his presence was urgently requested at the seat of government. Hastily Washington put his own affairs in order and then, just at nightfall, mounting his fleetest horse, he set out, not for the seat of government but to say good-by to his aged

mother. All through the hours of the night he rode and the next morning appeared unannounced at his mother's door. A brief visit and loving farewell, and Washington was on his way back to Mount Vernon. By nightfall he was again at home, having at the age of fifty-seven ridden more than eighty miles in twenty-four hours, over roads that were rough and primitive, for the sake of a last farewell with the mother to whom he owed so much. The next morning he was ready to start on his journey of two hundred and fifty miles to New York City.

We may well be thankful that in the most difficult moments in the history of our country we have had in the presidential chair men who believed in God and prayed to Him—Washington and Lincoln were men who under the most trying circumstances have kept their faith in the outcome of events clear and strong because they believed in God and carried their burdens to Him in prayer. We, who are citizens of the United States, may well try a little harder than ever before this year to be true to the high ideals set and the trust left to us by such leaders.

February: Fourth Sunday

SUGGESTED HYMN: "O Beautiful for Spacious Skies."
STORY:

JACOB A. RIIS

The acts of children are frequently the prophecies of the future. This was true at least in the case of one of our noted Americans who died a few years ago. At the age of twelve years, Jacob A. Riis, then a boy in Denmark, offered the money which had been given him for Christmas to a dweller in a slovenly tenement on condition that he would clean up his house and his children

No one realized then that Jacob would become an international character famed for his work in improving the condition of tenement dwellers.

Jacob Riis learned the carpenter trade, and at eighteen he asked the girl of his dreams to become his wife. She refused, and Denmark no longer had any attractions for Jacob. He wanted to get as far away as possible, and he started for America.

The experiences which Jacob had in trying to get established in the new world were severe, but they helped to make him a more useful man later. One night, in desperation, he applied at the police station for lodging. It was an experience never to be forgotten by Riis, for in the morning he saw a policeman, in a fit of rage, dash out the brains of a little dog which had been the wanderer's only friend. With rage such as he had never known, Riis hurled stones at the police station until the policemen drove him away. For years he carried the bitter memory of his unjust treatment in his heart. Many times he thought of vengeance. The years rolled around, and at last the opportunity for revenge came. He faced it squarely and turned from it. He would not seek personal satisfaction, but he would destroy the entire system of filthy and corrupt police lodging-houses in New York.' Progress was slow, but after many years Mr. Riis saw the last police lodging-house closed.

This, however, was only an incident in a much larger work. Gradually Mr. Riis came to give his entire time to the work of improving slum conditions. The famous Mulberry Bend, with its filth, its crime, its squalor, was forced to give way before his work, and a beautiful park took its place. He took upon himself the task of bringing cheer into thousands of homes which were cheerless

and of making living conditions more bearable for those who were ground under the heel of poverty. His books, "How the Other Half Lives," "The Battle with the Slums," "The Children of the Tenements," and others, have done more than perhaps any other books toward improving the condition of the poor in the cities of the United States.

In the midst of conditions which were disheartening in the extreme, Jacob A. Riis was able to maintain his optimism. The following words near the close of his life are typical of the man: "I have lived in the best of times. I have been very happy. No man ever had so good a time." By the pathway of service to his fellows the little Danish boy had become the famous American to whom Presidents of our republic were proud to do honor.

CHAPTER IX

WHAT IT MEANS TO BE A CHRISTIAN

March: First Sunday

This is the season of the year when many Sunday-school pupils are looking forward to joining the church. Themes which have to do with what it means to be a Christian will therefore be appropriate for the worship period.

ORDER OF SERVICE:

I MUSICAL PRELUDE:

II OPENING SENTENCES *(by leader or entire school)*

"I will praise the Lord at all times: His praise shall continually be in my mouth; O magnify the Lord with me, and let us exalt His name together."

III THE ONE HUNDREDTH PSALM: *(repeated in unison; school seated)*

IV HYMN: "God is Love" (*school standing*)

V STORY: (*This talk is designed to make the pupils realize that the Christian life is always a purposeful life.*)

DERELICTS AND OCEAN LINERS

Out on the Atlantic Ocean there are two kinds of boats. There is the great ocean liner with its chart, its compass,

its pilot, and its crew. It has its starting place, its definite course, and its destination. It has a purpose. There is another sort of boat, however. It has no chart, no compass, no pilot, no crew, no starting place, no course, no destination and no purpose. It is known as a derelict. It simply drifts. It is of no value to itself, and it is a menace to all who travel the seas. The United States Government is forced to spend thousands of dollars each year for the destruction of derelicts.

These two kinds of boats typify two kinds of lives. One is dominated by a great purpose, and makes every faculty bend toward the fulfillment of that purpose. The other drifts at the mercy of circumstances. The great religious men have always been men of purpose. We read that "Daniel purposed in his heart," and centuries later we remember Daniel with admiration. Paul was dominated by a great purpose. "This one thing I do," was characteristic of his whole nature. Martin Luther had a great purpose, from the accomplishment of which no danger could deter him.

Christianity comes into a life to give it purpose. It takes lives which would otherwise drift at the mercy of circumstances, and makes them of service to others. There are purposes which dominate lives besides the Christian purpose, such as the purpose to make money for selfish ends or to seek pleasure. All such purposes are more or less unworthy in character. Christianity alone furnishes a great unselfish purpose worthy of a son of God.

Every Sunday-school pupil must decide sooner or later whether he will be a derelict or an ocean liner—the first a menace to itself and to others, and the second a bless-

ing to the world because it has a purpose—and that purpose is to serve.

VI HYMN: "O Master, Let me Walk with Thee" *(school standing)*

VII PRAYER: *(by leader, or unison prayer)*

VIII RESPONSE: "Hear us, Heavenly Father, while on Thee we call. May Thy benediction on our spirits fall" *(or other appropriate response)*

March: Second Sunday

SUGGESTED HYMN: "Brightly Gleams Our Father's Mercy."

STORY: *(Purpose: To make the pupils more helpful to others, by suggesting that helpfulness is characteristic of the Christian.)*

THE HELPFUL HABIT

Jesus could always discover opportunities for helping others. In fact, that was one of his chief characteristics, as it has been of his closest followers throughout the ages. The best Christians have been those who carried good cheer and comfort with them as they passed along.

Phillips Brooks walked down the street one morning, and a newspaper man wrote, "The day was dark and rainy, but Phillips Brooks passed down Newspaper Row, and all was bright."

It is said that Henry Ward Beecher was walking one day down the streets of Brooklyn when he discovered a small boy sitting on the curb and crying as though his heart would break. The heart of the man was touched.

He picked the little fellow up in his strong arms and said, "What's the trouble, my little man?"

The boy looked through his tears and replied, "There ain't nothing the trouble, now you've come."

There are many places in the world where trouble could be eliminated if Christians took their task a little more seriously, and went about carrying the encouragement and the good cheer which is the Christian's by right.

Jesus taught that a man succeeds in proportion as he learns to give rather than to get. It is not always an easy lesson to learn.

Whether or not we really live the giving life depends a good deal upon what we are. If we make ourselves strong and joyous and kind, we shall inevitably be giving continually of our strength, our joy, and our kindliness, even when we are least conscious of it. The best things in life we cannot keep to ourselves even if we would. The first step in worth-while giving is to make ourselves worth while.

Over in India there grew on the grounds of a certain prince a mango of marvelous delicacy. Because the fruit was so fine, the prince said, "I will keep this entirely for myself."

In order to carry out his purpose, he built a high wall and placed soldiers about the tree to guard it night and day. Some years ago, however, a strong wind broke one of the branches from the tree, and it fell outside the wall. A pedestrian passing saw the twig in the path, carried it home and then, wrapping it carefully, sent it to a friend in Florida. The Florida friend nursed the little twig with great care; it grew into a tree and just recently it bore its first crop of forty mangoes. It is expected that before long there will be in the United States an

entire grove of mango trees of the same remarkable character as the single tree in India, which the prince is guarding so jealously for his own use and which he thinks he has kept entirely to himself.

The best things in life simply insist on being shared with others. It is our business to see that in our own natures we possess the best.

March: Third Sunday

Suggested Hymn: "Saviour, Like a Shepherd Lead Us."

Story: *(Purpose: To help the pupils understand the meaning of faith.)*

A CHILD'S FAITH

One of the things which should characterize a Christian is faith. Nothing can really harm one who trusts himself in the keeping of the God whom Jesus revealed. Sometimes we pray "Thy will be done" as though we were praying for the very worst thing which could possibly happen to us instead of the best.

A small boy had a pet kitten. It became necessary to dispose of the kitten. Ether was administered, and the boy sadly took his pet to the garden for burial.

Some time later the boy was on his way to the doctor's office, where a slight operation was to be performed. The mother explained that it would be necessary for the boy to take ether.

"But, mother," he said, "I don't want to take the ether."

"Mother thinks you had better take it."

"But I don't want to take it," said the boy.

"Mother wants you to," said the mother.

"All right," was the reply, "if you think it best, and you want me to take it, I will."

The two met the doctor. The ether was administered, and the boy was soon safely out from under its effects.

As he recovered consciousness, he looked up into his mother's face and said, "Why mother, I didn't die, and you didn't take me out and bury me like the kitten, did you?"

Then for the first time the mother realized that the boy had consented to take the ether because she thought it was best, when he believed that he would die and be buried as the kitten had been. Because he trusted her implicitly, he had put his life in her keeping.

This faith, so natural to the child, is not always easy to achieve, but it is the kind of trust which those who believe in the great Father-God of Jesus may have.

A Christian may have the faith of Job when he said, "Though He slay me, yet will I trust in Him"; or of Isaiah when he said, "Thou wilt keep him in perfect peace whose mind is stayed on Thee: because he trusteth in Thee."

March: Fourth Sunday

STORY: *(Purpose: To help the pupil to feel that in striving for the Christian ideal he may rely upon help from above.)*

THE MASTER PAINTER

Sometimes young people become discouraged in their attempts to lead the Christian life, because Christian standards are so high, and so difficult to attain. We must remember, however, that when we become Chris-

tians we have the finest and strongest forces of the universe working with us.

A painter was trying to copy the masterpiece of his teacher. Day after day, week after week, and month after month he worked. The outlines he could trace, and in many respects he made his painting look like that of his master. But the expressions of the faces and other fine points which made for the perfection of the original he could not reproduce.

At last, one day, tired and discouraged, he laid his head on the table and fell asleep. As he was sleeping the master himself came into the room. He saw what his pupil had been trying to do. He noted the places where he had failed. He saw the brush which had fallen from the hand of the sleeper. Picking up the brush, the master touched the picture here and there until it stood out just like the original. Quietly the master put aside the brush and left the room.

The pupil slept on. Finally he awoke, and looking up at his canvas he discovered that while he had been sleeping the picture had been finished.

It is in some such way, I suppose, that the Christian who does his best to attain the ideal which seems unattainable will find that he has achieved far more than he expected, because the great Master of all lives has been working with him.

March: Fifth Sunday

SUGGESTED HYMN: "Light of the World, We Hail Thee."

STORY: *(Presenting Christianity as a triumph over death.)*

TYLTYL AND MYTYL

If you have read the story of "The Bluebird" you will remember that there was once a little boy and a little girl named Tyltyl and Mytyl. One winter's night they were sound asleep when suddenly there appeared a very queer old woman demanding that the children secure for her "the grass that sings" or "the bluebird." The children had neither. The fairy, for such she proved to be, finally admitted that she could get along without the "grass that sings," but insisted that she must have "the bluebird." Accordingly she waved her wand, the old bedroom was transformed, familiar objects took on life and spoke, and the children started out upon their search for "the bluebird," which it is said always bring happiness to the one who possesses it.

In their search the children visit the Kingdoms of the Past, of the Future, and of the Dead, and the Realm of Night. It is while they are in the Kingdom of the Dead that a very interesting and significant incident occurs.

The children come timorously to an old country churchyard, where the moonlight falls on mossy slabs, sunken crosses, and neglected mounds. Little Mytyl is very much afraid, especially because her brother has told her that at the hour of midnight the dead leave their graves.

Mytyl wishes to run away, but Tyltyl, although frightened, insists on staying. At last the clock begins to strike. The children tremble. There is a moment of silence. The crosses totter, the mounds open, the slabs lift. They look for the dead, but no dead appear.

Instead, there arises gradually a blossoming of beauti-

ful white flowers filling and transforming the old cemetery into a fairy garden. Dew sparkles, flowers bloom, wind murmurs in the foliage, bees hum, birds appear and fill the place with their intoxicating song of life and sunshine and joy.

Amazed, the children hold tightly to each other's hands and look timidly among the flowers for some trace of graves, but no graves appear. Mytyl, searching among the grasses, asks, "Where are the Dead?" and Tyltyl, in his childish voice, cries out with all the earnestness born of a new discovery, "There are no dead!"

The children in their innocence had discovered the message which Easter has for you and for me and for all the world this year. It is a message that will bring cheer to millions of aching hearts to-day:

"There are no dead, for Jesus, on that first Easter morning, so long ago, triumphed over death, and henceforth its sting is gone for those who commit themselves to His keeping."

This is the message of Easter to us this Easter time.

CHAPTER X

THE EASTER MESSAGE

April: First Sunday

ORDER OF SERVICE:

I MUSICAL PRELUDE

II THE ONE HUNDREDTH PSALM: *(school standing and repeating in unison)*

III HYMN: "We March, We March to Victory."

IV PRAYER OF GOOD WILL: "Our Father in heaven, we thank thee that in work and in play, in joy and in sorrow, thou art the friend and companion of us all. When we do wrong and grieve thee, thou art ready to forgive. When we do right, thou art glad. May no hatred or envy dwell in our hearts. Keep our hands from selfish deeds and our lips from unkind words. Teach us to bring cheer to any who suffer and to share freely with those who are in need. So may we help thee, our Father, to bring peace, good will, and joy to all thy children. Amen."

(This prayer is taken from Hartshorne's *Book of Worship for the Church School.* Published by Charles Scribner's Sons. It is fine enough to be learned by the pupils and adopted as the school prayer.)

V HYMN: "Rejoice, Ye Pure in Heart."

VI STORY: *(The purpose of this story is to help the pupil to appreciate the significance of the Easter season and the Easter message.)*

HE ROSE AGAIN

A gentleman in one of our great cities stood looking at a picture in a store window. It was a picture of the crucifixion of Jesus. Suddenly he became aware that a street boy was standing by his side. "That's Jesus," said the boy. The man made no reply, and the boy continued, "Them's Roman soldiers," and, after a moment, "They killed him."

"Where did you learn that?" said the man.

"In a little mission Sunday school around the corner," was the reply.

The man turned and walked thoughtfully down the street. He had not gone far when he heard a youthful voice crying, "Say, Mister! Say, Mister!"

The gentleman turned to see his friend of the street hurrying toward him.

"Say, Mister," said the boy, "I wanted to tell you that he rose again."

That message, which was nearly forgotten by the boy, is the message which has been coming down through the ages. It is a message of Easter this year and every year, a message of the eternal triumph of life over death, a triumph which is continually being reënacted in the life of the Christian.

Many years ago a European princess died. That her body might never be disturbed, she arranged to have her grave covered with huge blocks of granite securely clamped together. Upon the grave this inscription was

placed: "This burial place, purchased to all eternity, must never be opened."

The months and the years passed, and the mandate inscribed in granite was respected. There was every indication that the wish of the dying princess would be carried out, and that her grave would forever remain closed. One day, however, a visitor noticed what seemed like a displacement of one of the large granite blocks. As the days passed this became more pronounced, until at last the granite gave way entirely, and a young oak tree stood forth, opening the grave which was to remain undisturbed through all eternity.

A tiny acorn had fallen all unnoticed into the grave, and what men dared not to do the acorn had done because it had in itself the germ of life. The seed of life, buried in the place of death, had brought forth of its own kind in spite of all efforts to suppress it.

Possibly no better illustration than this could be found of the way the Christian life operates. Christianity is not a set of rules or a cut-and-dried system to which all must conform. Rather, like the acorn, it is the germ of a new life coming to break up old traditions and habits and to remake the life from within.

VII PRAYER

VIII RESPONSE: *(by the school; an appropriate verse of a selected hymn sung softly)*

April: Second Sunday

SUGGESTED HYMN: "Christ the Lord Is Risen To-day."

STORY:

THE RESURRECTION

Now on the first day of the week cometh Mary Magdalene early, while it was yet dark, unto the tomb, and

seeth the stone taken away from the tomb. She runneth therefore, and cometh to Simon Peter, and to the other disciple whom Jesus loved, and saith unto them, They have taken away the Lord out of the tomb, and we know not where they have laid him. Peter therefore went forth, and the other disciple, and they went toward the tomb; and they ran both together: and the other disciple outran Peter, and came first to the tomb; and stooping and looking in, he seeth the linen cloths lying; yet entered he not in. Simon Peter therefore also cometh, following him, and entered into the tomb; and he beholdeth the linen cloths lying, and the napkin, that was upon his head, not lying with the linen cloths, but rolled up in a place by itself. Then entered in therefore the other disciple also, who came first to the tomb, and he saw, and believed. For as yet they knew not the scripture, that he must rise again from the dead. So the disciples went away again unto their own home.

But Mary was standing without at the tomb weeping; so, as she wept, she stooped and looked into the tomb; and she beholdeth two angels in white sitting, one at the head, and one at the feet, where the body of Jesus had lain. And they say unto her, Woman, why weepest thou? She saith unto them, Because they have taken away my Lord, and I know not where they have laid him. When she had thus said, she turned herself back, and beholdeth Jesus standing, and knew not that it was Jesus. Jesus saith unto her, Woman, why weepest thou? whom seekest thou? She, supposing him to be the gardener, saith unto him, Sir, if thou hast borne him hence, tell me where thou hast laid him, and I will take him away. Jesus saith unto her, Mary. She turneth herself, and saith unto him in Hebrew, Rabboni; which

is to say, Teacher. Jesus saith to her, Touch me not; for I am not yet ascended unto the Father; but go unto my brethern, and say to them, I ascend unto my Father and your Father and my God and your God.

April: Third Sunday

SUGGESTED HYMN: "May the Master Count on You?"
STORY: *(The particular purpose of this story is to increase the sense of responsibility on the part of those pupils who have united with the church at the Easter season.)*

JESUS' PLAN

Rev. S. D. Gordon, in one of his "Quiet Talks," allows his imagination to picture reverently what might have occurred when Jesus returned to be with the Father. It is somewhat as follows:

Gabriel says to Jesus, "So you have been down on the earth for a long time."

"Yes," replies Jesus.

"And you have lived and suffered and died for the people," says Gabriel.

"Yes," is the reply.

"And I suppose that every one knows about what you have done," continues Gabriel.

"No," answers Jesus.

"Well, what arrangements did you make for spreading the news and for carrying on the work which you have begun? Have you any plan?" queries Gabriel.

"Yes," replies Jesus; "I gathered a small group of followers and trained them as best I could. I lived with them and worked with them, and now they are

going out to tell others. Those others will tell still others, and so the work will go on until the whole wide world is encompassed."

"But suppose that the little group does not prove faithful," asks Gabriel; "suppose that Peter goes back to his fishing, that John gets discouraged, and that the others give up their work; have you any other plan?"

"No," says Jesus, "I have no other plan. I am depending on them."

This is the message which comes to us at the close of the Easter season. We are the emissaries of Jesus, doing his work and spreading the good news of him. If we fail, the work fails indeed, for he has no other plan. He is depending on us.

April: Fourth Sunday

SUGGESTED HYMN: "Faith of Our Fathers."

STORY: *(The purpose of this talk is to emphasize the importance of the times of quiet communion with God.)*

THE MINISTRY OF QUIET

A student of archæology not long ago discovered a very important inscription which had been dimmed by the ages. Naturally he was anxious to determine its meaning and significance, but he did not attempt to read the inscription at the time. Instead, he returned home and rested his eyes three days. At the end of that time he went back and read the inscription. He could not run the risk of reading inaccurately or of missing something in the inscription while his eyes were blurred with many other things.

An acquaintance of an artist was one day invited to

view a great new picture. Upon arrival at the artist's home the man was ushered into a partially darkened room and left alone for some time. He felt that he was not receiving the cordial welcome which he had expected. At last the artist appeared and explained that the quiet of the darkened room was a necessary preliminary in order that his visitor might get the glare of the street out of his eyes. Otherwise much of the beauty of the picture could never have been appreciated.

These two incidents are parables of significance for the Christian. The rush of events in our present-day life tends to make the Christian forget the times for quiet communion with himself and with his God; yet these times are as essential for the building up of strong Christian character as was the three-days' rest to the archæologist or the period in the darkened room to the artist's friend. We must take time regularly from the rush of events if we would "get the glare of the street out of our eyes" and discover some of the deep things of God which have been unappreciated by us. The habit of daily quiet communion with God will place the life of the Christian upon a sure foundation.

CHAPTER XI

THE CHRISTIAN AT WORK

May: First Sunday

The general purpose of the work of the month is to lead the pupils to understand that to serve one's fellowmen in need is to serve God. The leader's talk for the second Sunday of the month, however, is prepared especially for Mother's Day.

ORDER OF SERVICE:

I MUSICAL PRELUDE

II OPENING SENTENCES: "I will bless the Lord at all times; his praise shall continually be in my mouth. Oh, magnify the Lord with me, and let us exalt his name together."

III HYMN: "When Morning Gilds the Skies."

V Brief invocation by the leader, followed by the Lord's Prayer.

VI STORY:

"INASMUCH"

Jesus seemed to expect that his followers would go about the world doing some of the same sort of things which he did while he was on earth. Of him it was said, "He went about doing good." He comforted the broken-hearted, he relieved the distressed, he healed the sick, and he carried with him always the spirit of helpfulness and good cheer.

There is an old story of a woman who had heard that Jesus was to visit her town and that he would dine at the house which was best prepared to receive him. The woman determined that her house would be the one chosen by the Master. Accordingly she scrubbed and polished the house throughout and then set about the preparation of a wonderful meal.

Everything went well except that the woman had a number of interruptions as her work progressed. A plainly dressed caller tried to interest her in plans for improving the conditions of the needy in the community, but the woman replied, "I cannot give time to you to-day. The Master may come at any moment, and I must hasten my preparations."

A small boy playing in the street cut his finger and came to the door, asking for a cloth to cover the injured member. "Get away from here quickly," said the woman. "I have no time to hunt cloths to-day, and besides, you are getting blood on my steps."

The pastor called to ask for a few dollars for the needy ones in the foreign-mission field. "I believe in caring for the folks at home first," said the woman; "besides, I am very busy to-day getting ready for the coming of the Master."

There were other interruptions, and the woman at times became irritable. At last, however, the work was done, and she sat down in her spotless house to wait for the Master. All through the late afternoon and on into the night she waited, but the Master did not come.

The next morning, tired and dejected, she went about her daily task. As she worked her eyes fell upon the open book on the table and she read, "Inasmuch as ye did it unto one of these my brethren, even these

least, ye did it unto me." In an instant her mistake of the previous day was made clear to her. She had been so busy getting ready for the Master that when he had knocked at her door she had failed to recognize him.

Picking up the book, she read the words which were so familiar to her, but which this time carried a richer meaning than ever before. "Then shall the King say unto them on his right hand, Come, ye blessed of my Father, inherit the kingdom prepared for you from the foundation of the world; for I was hungry, and ye gave me to eat; I was thirsty, and ye gave me drink; I was a stranger, and ye took me in; naked, and ye clothed me; I was sick, and ye visited me; I was in prison, and ye came unto me. . . . Inasmuch as ye did it unto one of these my brethren, ye did it unto me."

VII HYMN: "O Master, Let Me Walk with Thee."
VIII PRAYER: *(by leader)*
IX RESPONSE: "Dear Lord and Father of Mankind."

May: Second Sunday

SUGGESTED HYMN: "There's a Wideness in God's Mercy."

STORY:

OUR MOTHERS

We observe Mothers' Day not that we may forget all about our mothers on the other three hundred and sixty-four days of the year, but, rather, that we may pause to say some of the things which are in our hearts every day but which do not often find opportunity for expression.

Some one has said, "God could not be everywhere and so he made mothers." Whether this is true or not, it is an attempt to say something which is true—namely, that a mother's love makes it easy for us to believe in a God of love. We know that God must be a loving God because he gave us our mothers. A Christian home presided over by a Christian mother is perhaps the greatest single blessing which can come into an individual's life.

Any one who has read the writings of Henry W. Grady, the famous Southern orator and journalist, has been struck with his sincere regard for the family relationships. He could hardly find words adequate to express his appreciation of the blessings of a Christian home. In attempting to make his viewpoint clear he relates how on one occasion he visited Washington, the capital of our country. As he gazed upon the magnificent buildings and as he saw the machinery of government in operation he said, "Here, indeed, lies the strength and greatness of our remarkable country."

Not long after this incident Grady was entertained in a humble country home in the South. There was an air of contentment about the place. Each member of the family had his allotted task during the day, and when evening came the father and mother and children gathered about the hearth, where they read a chapter from the Bible and then asked God's blessing upon the home before they retired. As Grady witnessed this scene he said: "I was mistaken the other day when I was in Washington. The strength of our great country lies not in piles of granite or in machinery of govern-

ment, but rather in homes like this, presided over by loving, God-fearing parents."

Some may not have understood why Grady never tired of praising the humble Christian home, but the reason was not far to seek. Grady himself was raised in just such a home by a mother who loved her children and her God. He tells us that at one time, after he was a grown man and bearing heavy burdens in the world, he became very much depressed; the burdens of life seemed too heavy, the perplexities too great; his faith in men seemed to be failing.

In such circumstances Grady slipped away from his task and from the great city of Atlanta. He went up to the modest country home where he had been born and where he had lived through his boyhood years. His gray-haired mother was still there to greet him. He sat by her side, and she told him the stories which he had heard so many times before. She prepared for him the same kind of food which he had enjoyed as a boy. At night she took him to his old room, and he said his evening prayer at her knee. After a few days of this life Grady went back to the great city. Again he felt brave, and his faith in God and in men was clear and strong because he had once more gotten into his life something of the spirit of his mother.

Such mothers are worthy of all the honor and consideration which we can give them this Mothers' Day and on every day.

May: Third Sunday

Suggested Hymn: "Take My Life and Let It Be."

Story:

A SOUTH SEA HERO

To live freely and gladly a life of service for others, until it becomes one's chief joy, is the privilege of the Christian. It is not always easy to give oneself in unselfish, sacrificial service to others, especially when that service does not seem to be appreciated. There are some, however, who, like their Master, are able to forget themselves in service even when opposition, rebuffs, and criticisms meet them at every turn. Such a man was James Chalmers, the missionary-hero of the South Sea Islands.

If there ever was a fun-loving, venturesome youth, it was Jim Chalmers. He was not the sort of a boy whom you would expect to become a foreign missionary, but when the serious question of deciding his lifework arose he gave himself to the foreign field.

He chose what was perhaps the most dangerous and difficult field in the world, the South Sea Islands. To the work there, among those barbarian peoples, Chalmers gave himself with zest. The climate was dangerous. The people were cannibals. Plots were laid; his life was threatened; he was attacked many times; he was shipwrecked several times and often in danger of drowning; he was stoned; and his drinking water was poisoned. His wife died, and he was left alone; yet when the London Missionary Society urged him to take a vacation from his work and come home, he wrote, "If I am to have a vacation I would prefer to spend it in opening up the interior of New Guinea."

No hardship was too great for Chalmers as he tried to help the ignorant and degraded savages among whom he worked to a better life. At last, however, after

twenty-one years of work Chalmers was induced to return to London for a time. It was while he was there that, in a speech before the directors of the society for which he worked, Chalmers revealed something of the joy which comes into the life of one who has learned the meaning of service to others.

"Recall the twenty-one years," he said, "give me back all its experiences, give me its shipwrecks, give me its standings in the face of death, give it me surrounded with savages, with spears and clubs flying about me, with the club knocking me to the ground—give it me back, and I will still be your missionary."

Chalmers had learned that the joy of serving others is richer and deeper than any other satisfaction which the world can offer.

On Easter Sunday, April 7, 1901, Chalmers's vessel anchored in front of a small island just off the coast of New Guinea. The next morning, amid canoes crowded with natives bearing spears, clubs, and knives, Chalmers started for the shore. His small craft entered a little bay, and there Chalmers disappeared forever from the view of men. A native afterward described what happened to the missionary when he and his young colleague, Oliver Tomkins, landed on that wild shore. He was struck down with a stone club and stabbed, his head was cut off, and his body cut into pieces and given to the women to be cooked and eaten.

Surely this was a cruel and revolting tragedy, but it was more than that—it was a glorious end to a noble life. Even Chalmers himself would not have had it different, for his death did much to break up the cannibalism of New Guinea.

A short time before Chalmers's death he wrote: "I

should not like to become a shelved missionary. Far better to go home from the field, busy at work."

The words contained in an address after his death expressed the feeling of thousands: "Know ye not that there is a prince and a great man fallen this day in Israel?"

May: Fourth Sunday

SUGGESTED HYMN: "When I Survey the Wondrous Cross."

STORY:

SAINT FRANCIS

About the year 1182 there was born in Italy in the home of a wealthy merchant a boy named Francis Bernardone. He was raised and educated in luxury. At a very early age he became the leader of a group of dissolute young spendthrifts in the community. His escapades were the grief of his mother and the talk of all the neighbors, but they were condoned by his father because his son mingled with the wealthy and aristocratic young set.

From one thing to another the young man went seeking satisfaction but never finding it. He knew that he lacked something essential to his happiness. He sought it in many places and, as a last resort, turned to religion. Filled with disgust at his own actions, he would slip away to some cave or grotto and there pray for hours. It was while he was in prayer one day that Francis caught a clear vision of the sacrifice and sufferings of Jesus and he realized that Jesus wanted his labor, his life, and all his being. He was filled with an overwhelming desire to share the burden which Jesus bore and to give his life in service to the needy.

Now, it chanced that in northern Italy at that time there were many needy. Beggars swarmed the roadside and the market place, and lepers and other unfortunates abounded. Francis had been used to luxury, but in the face of his new experience and of so much suffering he determined to abandon it all. To test himself he borrowed the rags of a beggar and all day long stood in a public place fasting with outstretched hand.

He left his home of comfort, gave away the possessions which were his own, and devoted himself to preaching and ministering to others in poverty. Where he could not give material aid he lavished his sympathy, and this was as much appreciated as his money. One day, at a turn in the road, he came face to face with a leper. The disease had always filled him with disgust and now he involuntarily turned away. Ashamed of himself, he turned back, gave the leper all the money he had, and then kissed the hand which received the gift.

Determined to follow the Master at all costs, and, incidentally, to conquer himself, Francis went again and again to the leper settlements, doing the most menial services for the unfortunates. He washed the sores of the distressed and on one occasion ate from the same porringer with a poor unfortunate who grieved that he had been cut off from his fellow men because of his disease; and, curiously enough, this youth, who had tasted all the pleasures of the world, found here in the service of the neediest of his fellows the joy which he had sought elsewhere in vain.

There is not time to tell of the many years which Francis spent in poverty and service. The life of this man has been written many times, and some day you

will want to read it. No longer is he known as Francis Bernardone, but he is called Saint Francis because he took his Christianity so seriously and gave his life in unselfish service for others.

CHAPTER XII

MAKING LIFE COUNT

June: First Sunday

June is a month of unusual significance to the young life in our Sunday schools. It is a time for reviewing the past months and for looking forward to the future. The spirit of commencement is in the air. Ideals are being crystallized. Whether the ideal is that of the selfish or the unselfish life will depend to some extent on the direction in which the thoughts of pupils are turned during the period of worship in the Sunday school.

ORDER OF SERVICE:

I MUSICAL PRELUDE

II OPENING SENTENCES: *(by leader)*

It is a good thing to give thanks unto the Lord. Oh, that men would praise the Lord for his goodness, and for his wonderful works to the children of men!

III HYMN: "My God, I Thank Thee" *(school to stand and remain standing until after the next hymn)*

IV THE ONE HUNDRED TWENTY-FIRST PSALM: *(repeated in unison)*

V HYMN: "We March, We March to Victory."

VI STORY: *(The purpose of this story is to empha-*

size in a concrete way the inadequacy of material possessions in the face of the great realities of life.)

"DOES A MAN NEED MUCH LAND?"

It is not an easy matter for young people to determine just what are the worth-while things in life. There is so much wealth in the world and sometimes we live under such abnormal conditions with so much luxury and convenience that we forget just what is and is not worth striving for.

Tolstoy, the Russian idealist, lived a very simple life and out of that life he produced some philosophy which is worth our consideration. Fortunately he has put much of it in story form. Under the heading "Does a Man Need Much Land?" or "How Much Land Does a Man Require?" he has given us a story which contains much food for the thought of young people.

A certain man was promised for his own all the land which he could walk around in one day. Delighted with the rich prospect before him, he arose before dawn and, taking his lunch in a pouch, started on his journey. Before the day had hardly begun he was well on his way. As the sun rose higher and higher it revealed fresh areas of beautiful land, and the man kept widening his circle, determined that nothing desirable should be omitted. Rapidly the hours slipped away, and faster and faster the man walked. At last the sun began to descend. It became necessary for the traveler to turn his steps toward the place of beginning. He was far from home, and every nerve must be strained if he would complete his circuit before nightfall. The race

between the rapidly sinking sun and the tired but determined walker became fast and furious. It was a question which would win. By a burst of speed, however, the man completed his circle just as the sun was slipping behind the horizon. He had reached his goal, and the land he had coveted was his. As he finished the circle he paused, grew faint, then tumbled over, dead from exhaustion. He was buried in six feet of earth, and Tolstoy leaves us to consider the question, "Does a man need much land!"

The parable of the Russian idealist is, after all, a parable of life, and its message is one well worth thinking over.

VII PRAYER

VIII RESPONSE: *(first stanza of* "Dear Lord and Father of Mankind.")

June: Second Sunday

SUGGESTED HYMN: "True-Hearted, Whole-Hearted."

STORY: *(This story is designed to reënforce the idea presented last Sunday.)*

THE MAGIC SKIN

Young people are always desiring pleasure and are often keenly disappointed when it does not come to them in full measure. For all the pleasures we get out of life, however, a price is demanded. One of the important questions every young person should ask himself again and again is, "Is this satisfaction worth the price asked for it?" Many of the things we think we most desire can be purchased only at a price that makes them undesirable.

There is an ancient fable concerning a skin which one might put on and secure everything that was desired. The only embarrassing thing in connection with the process was the fact that every time one wished for a selfish thing the skin itself grew smaller, until at last, just as every selfish desire had been gratified, it choked out the life of the wisher.

There are few things in life which one may not have if he is willing to pay the price. In too many cases, however, the acquisition of material blessings, together with the opportunity for the unlimited gratification of the appetites, narrows the sympathies and dwarfs all that is finest and most godlike in us.

There is a firmly grounded idea in the minds of many young people that being a Christian eventually involves the sacrifice of some of the greatest pleasures in life. Although, as a matter of fact, this idea is built upon a false foundation, it is true that Christianity involves sacrifice. We should, however, consider in the same connection some related facts.

Christianity involves sacrifice, but so does business, art, politics, and the professions. There is nothing worth while to be attained in life which is not purchased at the cost of something else. A man who would succeed in business must be prepared to give up indolence, ease, and personal comfort and give himself unreservedly to the task in hand. Arduous labor and oftentimes extreme physical hardships are involved, but if the end sought is deemed worthy, the price of success is never considered too high.

Sacrifice, then, is not peculiar to Christianity; it is a law of life. One may not avoid the necessity for sacrifice by refusing to become a Christian. Instead Chris-

tianity is the one guarantee that sacrifice shall not be in vain. Many a man has sacrificed the finest things in life for money or honor or position, only to find that he has made a poor bargain; that he has squandered gold for dross. No sacrifice which Christianity involves is ever in vain, for Christianity includes all that is most deep and real of peace and joy and success.

We may feel that in choosing the Christian life we choose to give up many things, but no sooner is the choice made than we discover that we have instead chosen for ourselves all the joys and satisfactions which a loving Father God could devise for his children.

June: Third Sunday

SUGGESTED HYMN: "O Master, Let Me Walk with Thee."

STORY: *(The purpose of this story is to inspire the pupils to seize every opportunity for serving their fellow men.)*

THE HOUSE BY THE SIDE OF THE ROAD

A poet was walking one day from one town to another along a country road. The day was hot, and the walker was tired. As he passed under the shade of an overhanging tree, he noticed a wooden bench and, fastened to the bench, a rough sign, "Sit Down and Rest."

Accepting the silent invitation, the poet sat down. Near by was a basket filled with beautiful early apples, and again he saw a small sign, "Have an Apple." The traveler ate an apple and, as he looked about, he saw a card on a tree, "At the End of This Path Is a Cool Spring." The poet was thirsty, and the mystery of it

all attracted him. He followed the path, and there he found the promised spring and a cup. By this time the interest of the poet was keen. He saw not far away a small cottage and coming from it an old man with a very kindly face.

Pressed by the questions of the traveler, the old man explained half apologetically: "Well, you see, a good many people walk past this way, and often the days are hot. We weren't using the old bench very much and we thought it would give some who were tired a chance to rest in the shade. Then when the apples got ripe, we had more than we could use, so we have kept the basket filled by the roadway. Then it occurred to us that the spring was hidden by the bushes, and that some might not know just where to find it, so we put up the sign to call attention to the spring."

The poet thanked the old man, and then as, refreshed and inspired, he continued his journey, a poem formed itself in his mind. The following quotation from the poem gives us its message:

"I would live in a house by the side of the road,
Where the race of men go by,
The men who are good, the men who are bad,
As good and as bad as I.
I would not sit in the scorner's seat,
Nor hurl the cynic's ban.
I would live in a house by the side of the road,
And be a friend to man."

June: Fourth Sunday

SUGGESTED HYMN: "Who Is on the Lord's Side?"

TALK

THE MAN WHO COMES UP FROM THE CROWD

Young people sometimes get the idea that all the big tasks of the world were done in the past. All the great discoveries seem to have been made, all the great inventions completed, all the great fortunes amassed, all the great books written. All the Pauls and the Luthers and the Abraham Lincolns seem to have had their day. Now there would appear to be nothing for us to do but to accept the world as we find it and to follow along the paths laid out in the past. As a matter of fact, no conception could be further from the truth than the one just indicated.

Never were there so many big tasks to be done as now. There are scores of millions of boys and girls in China without schools of any sort who need to be taught. There are other millions in India waiting and begging to be told the facts about Jesus Christ and the message which he came to deliver. There are uncounted tribes in Africa who have never heard of Jesus Christ. There are multitudes in South America and Mexico who cannot read or write and who lack the purifying touch of true Christianity. There are twenty-seven million boys and girls and young people in the United States under twenty-five years of age who are not enrolled in any church school, Protestant, Catholic or Hebrew. There are newcomers to America who have never had a chance to know a Christian American and who are still waiting for the touch of a friendly hand in the land which they have chosen to make their home. There are social wrongs in abundance which must be righted before any young person can choose with a free conscience a life of selfish indolence or ease.

Yes, great deeds have been done in the past, but greater ones must be done in the future if the religion of Jesus Christ is to be advanced in the world. Noble, unselfish lives have been lived, but never were the demands for unselfish service louder than now. There are problems before the church to-day which are as great or greater than any which it has ever faced. Will these problems be solved? Will the unselfish lives be lived? Will the great deeds be done?

All of these questions will be answered "yes" or "no" by the pupils now in our Sunday schools. There is no one else to provide the answer. If the answer is "yes" it will be because there are some boys and girls who are willing to do the hard thing, the unusual thing, the thing that demands the great sacrifice, the thing that may seem to involve doing more than our fair share of the world's work.

In the days of the world war young men and young women unselfishly surrendered positions, opportunities, personal comforts and friends that they might serve to the uttermost. Now the war is over but the great cause to which these young people gave themselves is not won. It must be won by the same spirit which, during the war, led so many to step out from the ordinary walks of life to do the unusual and the hard thing.

"Men seem as alike as the leaves on the trees,
As alike as the bees in a swarming of bees;
And we look at the millions that make up the state,
All equally little and equally great,
And the pride of our courage is cowed.
Then Fate calls for a man who is larger than men.
There's a surge in the crowd—there's a movement—and then
There arises a man that is larger than men—
And the man comes up from the crowd."

It makes us wonder how many there are in our own Sunday school of whom it will be said, "He was different from the common crowd, because in a time of very unusual opportunity he forgot himself and did the great unselfish deed which so much needed to be done."

CHAPTER XIII

THE WONDERS OF GOD'S WORLD

July: First Sunday

The summer season has its peculiar religious message for pupils of the church school. The service of worship should be planned with this thought in mind. The purpose underlying the following programs is to make God more real to the pupils by opening their eyes to the wonders of the world about them.

ORDER OF SERVICE

I MUSICAL PRELUDE

II HYMN: " From All That Dwell Below the Skies."

III OPENING SENTENCES: "The earth is the Lord's, and the fullness thereof. My soul shall make her boast in the Lord; the humble shall hear thereof, and be glad. O magnify the Lord with me, and let us exalt his name together."

IV HYMN: "We Plow the Fields and Scatter."

V PSALMS: (*repeated in unison*) The One-Hundredth or the Twenty-Fourth.

VI STORY:

THE TREES

A long time ago the poet wrote:

> "To him who in the love of Nature holds
> Communion with her visible forms, she speaks
> A various language."

One of the perennial joys of the summer is the opportunity which it brings for communion with nature and with nature's God. The beauty of nature stirs us to thanksgiving; its vastness fills us with reverence and awe; its adaptations to our needs convince us anew of the loving care of a Father God. From the study of nature we learn parables which bring to us moral lessons and we discover ways by which we can coöperate with nature in carrying out the great purpose of God.

Among all the works of God in nature perhaps none is more wonderful than the trees. For centuries they have been the objects of admiration and of wonder. Poets and orators have found in them an inexhaustible storehouse of fact and suggestion of which the mind of man seems never to tire.

Down in southern Mexico there stands a giant cypress tree approximately one hundred and twenty feet in circumference. It is said by many to be the oldest living thing on the face of the earth, even outranking in age the famous General Sherman tree in California. This cypress tree was several thousand years old when Jesus was born. It is as old as Archbishop Usher believed the human race to be. In its body it bears the marks not only of centuries, but of all the centuries since the history of man began to be written. Its trunk is indented and gnarled, yet it has withstood the attacks of man and of the elements and to-day stands out vigorous and strong. Its branches are healthy, and there is no sign of disease or decay among its members.

This tree brings to us to-day out of the past a lesson in optimism and faith. Surely no one could have seen the wonderful possibilities for growth bound up in the little sapling which began its career so many thousand

years ago. The tree has not grown to its present size or grandeur all in a minute, but steadily, year after year, it has added a little new wood and a few new branches until it has become the giant of the earth.

Thus it is that we attain to the worth-while in life, not because we see the end from the beginning and not by a single leap, but by steadily and patiently making use of the opportunity which lies just ahead. Then, too, the great tree makes it easier to believe in the eternity of God. Surely, if he can keep a tree through all the vicissitudes of six thousand years, he can care for us in spite of the great mystery of death.

VII HYMN: "My God, I Thank Thee."
VIII PRAYER
IX RESPONSE: "Dear Lord and Father of Mankind."

July: Second Sunday

SUGGESTED HYMN: "There's Not a Bird with Lonely Nest."

STORY:

THE BIRDS

The interrelation of the various parts of nature furnishes a suggestive field for investigation. Last week we were thinking of the trees—their beauty, their service to man. Surely they are one of God's great blessings; yet even this blessing is dependent on still another. Those who have studied the matter most carefully tell us that without the birds we could not have the trees. The enemies of the trees are many. If these enemies were allowed to prosper unmolested, the trees would be doomed. The trees furnish the homes for the birds, and

in return the birds rid the trees of their insect pests.

Possibly no bird does more for the trees than does the woodpecker. Freeing the trees of the pests which attack them internally is the particular task of the woodpecker. We may understand how important this work is when we pause to realize how numerous the enemies of the trees are. More than four hundred kinds of insects are known to prey upon the oak, and many of these multiply with great rapidity when left unmolested. A single insect borer may stunt or even kill a tree. One woodpecker may save hundreds of trees from damage or death in a single season. One bird has been known to inspect eight hundred trees in a single day. At other times a woodpecker has spent several days upon the trunk of one badly infested tree.

Our birds, however, do not confine themselves to fighting the enemies of trees. They help the farmer by eating each year millions of weed seeds and, when given a chance, they destroy the enemies of the farmer's crops. Cotton-growers are said to lose one hundred million dollars a year because of the wanton destruction by hunters of quail, prairie chickens, meadow larks, and other birds which feed upon the boll weevil.

These gifts of God are more than mere ornaments. They, too, have their part to play in making the world a fit place in which to live.

July: Third Sunday

SUGGESTED HYMN: "All Things Bright and Beautiful."

STORY:

THE FLOWERS

"Flower in the crannied wall,
  I pluck you out of the crannies,

> I hold you here, root and all, in my hand,
> Little flower—but if I could understand
> What you are, root and all, and all in all,
>   I should know what God and man is."

God sends his flowers in such profusion that to the thoughtless of his children they sometimes seem common. The miracle of even the commonest flower is, however, unfathomable, and it is only our blindness that keeps us from the proper appreciation of it. It often takes some unusual experience to open our eyes. Such an experience a certain Frenchman had many years ago.

This man had been unjustly imprisoned, and as he trod the dirt flood of his cell his heart was filled with bitter thoughts about God and man. On the walls of his cell he wrote, "All things come by chance." Lonely, miserable and rebellious, the prisoner spent his days. He had nothing to care for and nothing to do but think. Nothing unusual had ever happened during his imprisonment. Each day the routine of the previous day was repeated, until one day an event occurred which was destined to alter the fate of the unfortunate victim. Close to the wall of the prison, where the dirt was not quite so hard as elsewhere, a tiny green shoot thrust its head above the ground. The man, who without a tremor, had looked upon thousands of acres of green, was tremendously stirred by the appearance of this single tiny shoot. Here was something alive and growing; here was companionship; here was something to love and tend.

Very gently did the man stir the ground around the young shoot and with jealous care he watched it develop day by day. Slowly it grew larger and larger. A bud appeared; and then, as if for the individual enjoy-

ment of the prisoner, it burst into a wonderful pure white flower. Already the man had ceased to think of himself. His thoughts were ever of the flower and then of the God who had sent it. One day he read the words which he had written upon the wall of his prison and he felt ashamed. He rubbed them out and then he wrote in their place: "All things come from God."

Finally the story of the flower spread abroad and after a time came to the ears of the queen. The queen became interested and after some effort secured the release of the prisoner; "for," said she, "a man who can love a flower like that is surely not all bad."

July: Fourth Sunday

SUGGESTED HYMN: "Summer Suns Are Glowing."
STORY:

THE ANIMALS

We live in an age of cities and of machinery. The automobile is displacing the horse; mouse traps, with their increased efficiency, are eliminating cats; policemen have replaced watchdogs; the necessities of city sanitation have made pets undesirable; even the wild life is driven out by the very buildings of the city itself. Gradually the American family is finding itself independent of the animal creation and is dispensing with it, or, at least, rendering contacts with it more and more indirect. In all this change there lurks a very real danger—the danger that the present generation of boys and girls will fail to learn the lessons which, for untold generations of God, has been teaching through the animals.

The wonderful ways which nature devises for meet-

ing the needs of her creatures, the adaptation to environment, the compensation for handicaps, and the protection against enemies, are most suggestive of the ever watchful care of the heavenly Father for his children. The thickening of the coat of the horse and the dog as winter approaches and the shedding of the same coat when it is no longer needed would fill us with amazement were we not so accustomed to it. The soft-padded foot of the cat, which enables her to walk so quietly, and the sharp claws, which appear when she wishes to grasp a victim for food or when she must climb a tree to escape some enemy, are all evidences of adaptation to the necessities of life. The deer has little ability to defend itself, but it is made fleet of foot, so that it can outdistance its enemies. As a still further protection it is colored like the foliage among which it lives. The horse must travel rapidly over hard and often very rough places. Its hard hoofs enable it to bear its heavy weight without injury. The camel lives on the desert and is given a pad to keep from sinking into the soft sand. The long legs of the giraffe made it difficult for him to reach the ground with his head, but he is assured an abundance of food by being given a neck so long that he is able to reach much higher than any other animal. To compensate the elephant for his short neck he is given a trunk which enables him to reach the ground with ease.

The foregoing are only suggestions of the way nature provides for the meeting of emergencies in the lives of animals and for every apparent deficiency in equipment furnishes an adequate compensation. Surely a loving God will not do less than this for his highest creation, man.

This is only one of many lessons which the animals will teach us this summer if our ears are alert to their message.

July: Fifth Sunday

SUGGESTED HYMN: "The Summer Days."
STORY:

THE INSECTS

There is one entire realm of God's creatures which we rarely consider unless we are forced to give attention to it. In spite of this fact, however, this realm, the insect world, comprises by far the largest group of living creatures of the world. The number of individuals is countless, while the number of species is estimated at more than five million.

Ordinarily we think of insects as a nuisance and we would not be seriously concerned if we heard that they were all to be exterminated. Often their annoying attacks upon us make them seem the one blot upon the summer. It adds further to our personal irritation when we are told by the United States Department of Agriculture that each year the insects destroy more property than was represented by the annual expenditure of the United States for the National Government, the pension roll, and the maintenance of the army and navy before the war. The Hessian fly ruins annually seventy-five million dollars' worth of grain. The chinch bug levies a tax of thirty million dollars on corn. Hay is reduced ten per cent by the army worm, while cotton, fruit, vegetables, timber, livestock, and even manufactured articles pay a toll to the army of insects, which is astounding.

Although injurious insects destroy much fruit and grain each year, yet without other insects we would

have no fruit at all. Practically all our fruit and all our most beautiful flowers are dependent on insects for pollination.

Insects are responsible for the rapid disappearance of decaying vegetable and animal matter on the earth. Insects provide us with honey, silk, and other articles of commerce. Insects, too, are the only agents which will keep injurious insects in control. In fact, we are told by those who know most about insects that their service is so great that without them the earth would be uninhabitable in a very short time.

As we think of the insects whose depredations are so many and serious, and yet whose services are so great that man is dependent upon them for existence, does it not suggest that for all of the limitations and sorrows of life there may be compensations in terms of a deeper, richer life, which converts even our trials into blessings?

CHAPTER XIV

OUR GREAT HYMNS

August: First Sunday

The fundamental purpose of the leader's talks for the month is to assist the pupil in worship by permanently enriching his knowledge of our great hymns. Incidentally certain other ends will be served as the work of the month progresses. (For further details of these and other hymns see *A Treasure of Hymns,* by Amos R. Wells.)

ORDER OF SERVICE

I MUSICAL PRELUDE

II OPENING SENTENCES:

"Make a joyful noise unto Jehovah, all ye lands.
Serve Jehovah with gladness:
Come before his presence with singing.
Sing unto him, sing praises unto him;
Talk ye of all his marvelous works."

III HYMN: "O Day of Rest and Gladness."
(School stands and remains standing for the Psalm and the following hymn.)

IV THE ONE-HUNDREDTH PSALM: *(Repeated in unison. The use of this Psalm, as of others, should be announced one or two weeks in*

advance, so that those who do not already know it may learn it.)

V Hymn: "We Come with Songs of Gladness."

VI The Lord's Prayer: *(Or other unison prayer. Seated with bowed heads.)*

VII Story:

SARAH FLOWER ADAMS

Just as we know a child better when we know his parents, so we understand a poem or hymn clearly when we know its author. A single meeting with a great author has sometimes lifted a whole life to a new level of interest. We may not all have the privilege of meeting authors, but we may know the stories of their lives and work.

Sarah Flower Adams is not the best known of women hymn writers, but she is credited with writing the greatest hymn ever written by a woman. The romance of her story began even before she was born, for her father and mother first met while he was a prisoner in a London prison. His offense was the serious one at that time of holding liberal ideas in politics. It was here that Miss Eliza Gould, a sympathizer with his views, visited him, and upon his release they were married.

Sarah was born February 22, 1805. Her mother died when Sarah was five years of age, and she was left with one sister, Eliza. Sarah was a poet by nature, and Eliza was musical, so Eliza provided the music for the hymns which Sarah wrote.

In 1834 Sarah married a civil engineer. She was beautiful, of delightful manners and conversation, and of exalted character. Even after her marriage Sarah and her sister were much together. Both died young,

and the hymns sung at both funerals were by Sarah with music by Eliza.

Sarah's greatest hymn, which we are to sing this morning, is "Nearer, my God, to Thee." It became popular in America about 1856. In 1872 it was sung at the Boston Peace Jubilee by nearly fifty thousand voices.

Numerous interesting incidents have been connected with the use of this hymn. During the Johnstown flood in 1889 a young lady on her way to the foreign-mission field was imprisoned in her car beyond hope of rescue, and as the waters rose about her those near by heard her voice as she sang "Nearer, My God, to Thee."

As President McKinley was dying those at his side heard him murmur, "'Nearer, my God, to thee, e'en though it be a cross,' has been my constant prayer." On the day of his funeral, September 19, 1901, at half-past three o'clock, people all over the land paused and waited in silence. Cars were stopped; street traffic ceased; and men stood with bared heads. For five minutes there was silence throughout the land. At the close of this period bands in various squares in New York City played, "Nearer, My God, to Thee," and the same hymn was used in countless memorial services held in honor of our martyred president.

Such are some of the associations connected with the hymn which we are to sing at this time:

"Nearer, my God, to thee,
  Nearer to thee!
E'en though it be a cross
  That raiseth me;
Still all my song shall be,
Nearer, my God, to thee,
  Nearer to thee!"

VIII Hymn: "Nearer, My God, to Thee."
IX Prayer: *(by leader)*
X Response: *(by school or choir:* "Immortal Love," *first stanza only)*

August: Second Sunday

Suggested Hymn: "Speed Away."
Story:

FANNY CROSBY

Of all the writers of hymns perhaps the one best known to Americans is Fanny Crosby, the greatly-loved, blind hymn-writer who died only a few years ago. Thousands of persons all over the United States had the opportunity of shaking her hand and listening to her kindly words of cheer and wisdom.

She was born in New York State in 1820, and her life lacked only a few years of covering an entire century. When only a few months of age she became blind, so that she could never remember seeing the sunlight or the beautiful world about her. In spite of that fact she was always contented, cheerful, and optimistic. When only eight years of age she wrote:

"Oh, what a happy soul I am!
  Although I cannot see
I am resolved that in this world
  Contented I will be;
How many blessings I enjoy
  That other people don't!
To weep and sigh because I'm blind
  I cannot and I won't."

At a proper age Fanny Crosby was sent to a school for the blind in New York. Here she graduated and afterward became a teacher. While still a pupil she re-

cited a poem before the Senate and the House of Representatives in Washington as an illustration of what education might do for the blind.

Although she became a teacher and afterward was married, her real lifework was the writing of songs and hymns, of which she wrote more than three thousand. "There's Music in the Air" is one of her best known secular songs, while her sacred songs are almost numberless. "Jesus, Keep Me Near the Cross," "'Tis the Blessed Hour of Prayer," "Rescue the Perishing," "Pass Me Not, O Gentle Saviour," "Safe in the Arms of Jesus," "Some Day the Silver Cord Will Break," "Blessed Assurance, Jesus is Mine," "Saviour, More Than Life to Me," and "All the Way My Saviour Leads Me" are samples of her work.

Fanny Crosby wrote rapidly. Some of her best hymns seemed to come as an inspiration, with little effort on her part. She had a remarkable memory, which was of great help to her. When she was a child she committed to memory the first four books of the Old Testament and the four Gospels.

Some one told Fanny Crosby one day that a man on a western prairie had been attracted by the words of one of her hymns to a meeting which was in progress in a very primitive building near by. He had entered the building and heard a message which transformed his life. When Miss Crosby heard the story, she said, "If that is true, it pays me for a whole life of effort, just to know that I have had a little part in the transformation of one person."

The hymn that she wrote which we are to sing to-day has done much for the foreign-mission cause. It has become the farewell hymn for departing missionaries

and is regularly used as they set out for their fields of labor:

"Speed away! speed away! on your mission of light
To the lands that are lying in darkness and night;
'Tis the Master's command; go ye forth in his name,
The wonderful gospel of Jesus proclaim;
Take your lives in your hand, to the work while 'tis day,
Speed away! speed away! speed away!"

August: Third Sunday

SUGGESTED HYMN: "My Faith Looks Up to Thee."
STORY:

"MY FAITH LOOKS UP TO THEE"

Two weeks ago we spoke about and sang the hymn which is said by some to be the greatest hymn ever written by a woman. To-day we are to consider a hymn which is perhaps the greatest one ever written by an American. Its author, Ray Palmer, was born in Rhode Island in the year 1808. He became a clerk in a Boston dry goods store and later a student at Yale. He taught school in New York and New Haven, entered the ministry and held several important pastorates and one secretarial position. He lived to be nearly eighty years old, and his life was filled with worth-while work. He is best remembered, however, as a writer of hymns, and his most famous hymn, "My Faith Looks Up to Thee," was written when he was only twenty-two years of age.

At the time the hymn was written Mr. Palmer, afterward Dr. Palmer, was teaching school in New York City. The verses represented the spiritual experience of an earnest and devout soul. They were written in a small pocket notebook for the author's use in his hours of communion with God. Two years later Palmer was

in Boston and by what seemed to be the merest chance met on the street Lowell Mason, the famous musician. Mason wanted a hymn for a publication which he was getting out at the time. The two men went into a store and there copied "My Faith Looks Up to Thee," which had been written two years before.

Mason took the hymn and wrote for it the tune "Olivet," to which it has always been sung. Shortly after Mason again met Palmer. On this occasion Mason said, "You may live many years and do many good things, but I think you will be best known to posterity as the author of 'My Faith Looks Up to Thee.'" The words have proved true, for, while Palmer wrote many fine hymns, this, his first production, is the grandest of them all.

Palmer was a most admirable and lovable man and a man of deep feelings. He tells us that he was so affected by the writing of his own hymn that he burst into tears when it was completed. He says of it: "It was born of my soul." Surely a hymn of such beauty thus conceived will not soon die. It has been translated and used in thirty different languages, and it has thus become a great world hymn of the church.

"My faith looks up to thee,
Thou Lamb of Calvary,
Saviour divine!
Now hear me while I pray,
Take all my guilt away,
Oh, let me from this day
Be wholly thine."

August: Fourth Sunday

Suggested Hymn: "How Firm a Foundation."

Story:

"HOW FIRM A FOUNDATION"

Occasionally a famous old hymn comes down to us without the name of the author. This is the case with "How Firm a Foundation!" It has been attributed to several different authors, but no one seems to know for sure just who wrote it. It is now supposed to have been written by a man named Robert Keene. It was first published about one hundred and thirty years ago, and it has been in constant use ever since.

Frances Willard related that it was used at family prayers in her mother's home and that it had been similarly used by her grandmother and her great-grandmother. Thus, in one family this fine hymn had served the religious needs of four generations.

The hymn was widely used by both armies during the Civil War. It was the favorite hymn of the famous general, Robert E. Lee, and was sung at the great commander's funeral. It was also sung by request for former President Andrew Jackson in his old age. He remembered it as the favorite hymn of his departed wife.

On Christmas Eve, 1898, the Seventh United States Army Corps was encamped on the hills above Havana, Cuba. At twelve o'clock a sentinel from the Forty-ninth Iowa Regiment began to sing "How Firm a Foundation!" Other voices joined until the whole regiment was singing. Then a Missouri regiment added its voices. The Fourth Virginia followed, and all the others until, as General Guild said, "On the long ridges above the city whence Spanish tyranny once went forth to enslave the New World, a whole American army corps, Protestant and Catholic, South and North, was singing:

"'Fear not; I am with thee; oh, be not dismayed;
For I am thy God, and will still give the aid;
I'll strengthen thee, help thee, and cause thee to stand
Upheld by my righteous omnipotent hand.'"

August: Fifth Sunday

SUGGESTED HYMN: "My Country, 'Tis of Thee."
STORY:

"MY COUNTRY, 'TIS OF THEE"

Every child knows "My Country, 'Tis of Thee," and that it breathes a fine religious sentiment. It is not so well known, however, that this hymn was written by a Christian minister. Possibly it would be more correct to say that it was written by a young man preparing for the ministry, for Samuel Francis Smith was still in school when he wrote this hymn.

He tells us that one dismal day in February, 1832, he was turning over the leaves of a music book when his eyes rested upon the tune to which "America" is now sung. He did not at that time know that it was the music of "God Save the King." The idea of writing a patriotic hymn to fit the music came to him, and he sat down and within about thirty minutes wrote the hymn just as it stands to-day. It was originally written on a scrap of waste paper five or six inches long and two and one-half inches wide.

Dr. Smith said of it later: "I never designed it for a national hymn and I never supposed that I was writing one."

The hymn was first sung at a children's celebration in Boston in 1832. It has never been adopted by our Government as a national anthem, but it has been adopted by the people. Through it for nearly a century the peo-

ple of the United States have voiced their love for their country and their sense of dependence on and trust in God.

The fact that the music of "America" is that of "God Save the King" has given rise to many impressive scenes. At international religious gatherings one stanza of each hymn has sometimes been sung and this followed by "Blest Be the Tie That Binds."

Dr. Smith wrote many other hymns, among them the famous missionary hymn, "The Morning Light Is Breaking," but he will be best remembered for the hymn which, as a student, he wrote in a few minutes on a stormy day in February and which, without any intention on his part, has become one of our national hymns:

"My country, 'tis of thee,
Sweet land of liberty,
Of thee I sing:
Land where my fathers died,
Land of the pilgrim's pride,
From every mountain side
Let freedom ring!"

CHAPTER XV

THE BIBLE MEETING THE WORLD'S NEEDS

September: First Sunday

ORDER OF SERVICE:

I MUSICAL PRELUDE:

II HYMN: "Praise God from Whom All Blessings Flow." *(Sung by school choir or by the entire school without announcement.)*

III OPENING SENTENCE: "The earth is the Lord's, and the fullness thereof; the world, and they that dwell therein. Oh, that men would praise the Lord for His goodness, and for His wonderful works to the children of men!"

IV HYMN: "O Worship the King." *(School stands and remains standing for the Psalm and the following hymn.)*

V THE TWENTY-THIRD PSALM *(in unison.)*

VI HYMN: "O Word of God Incarnate."

VII STORY:

DISTRIBUTING THE BIBLE

A boy born into a home supplied with wholesome running water may grow into manhood without once experiencing extreme thirst or having any appreciable sense

of gratitude for the blessing which is his even without the asking. A person who has been lost in such a desert as Death Valley in California knows more about thirst and the value of water than ten thousand boys raised in city homes. Thus it is with most of our blessings, including the Bible. We have been told many times that the Bible is the greatest book in the world and that we should be very grateful for it, but the fact that we have never been deprived of the Bible tends to make us take it as a matter of course.

A little more than one hundred years ago in Wales a certain minister asked a little girl if she could repeat the text of his sermon the previous Sunday. Instead of making reply, she remained silent for a moment and then began to weep, saying, "The weather has been so bad that I could not get to read the Bible during the past week." The little girl had been accustomed to travel seven miles over the mountains each week to a place where she could get access to a Bible.

This scarcity of Bibles so impressed the minister, Thomas Charles, that it resulted in the formation of the British and Foreign Bible Society. This society alone has printed and distributed up to date more than two hundred and forty-five million copies of the entire Bible or parts of it.

It is interesting to know that the first Bible printed in the United States was printed in the language of the American Indians from a translation made by the famous Indian missionary, John Eliot, in 1663. In 1734 a German Bible was printed in Germantown, Pennsylvania, and in 1782 the first English Bible in America was printed in Philadelphia. Bibles were so scarce during the Revolutionary War that Congress ordered twenty

thousand copies of the English Bible imported at public expense.

Some years later the American Bible Society was organized. Since its organization it has distributed more than one hundred million copies of the Bible or parts of it. Four different times in its history it has undertaken to canvass the entire United States and to place a Bible in every home. To-day a Bible may be had by any one for the cost of printing, and if this price is prohibitive, the Book will be given without charge.

All of this effort to place the Bible in the hands of every one has cost years of labor and much sacrifice on the part of many. Surely this thought alone ought to make us appreciate more than ever the lessons which we study from week to week in our church school.

VIII HYMN: "Father, Again to Thy Dear Name We Raise."

IX PRAYERS *(in unison.)*

X RESPONSE: *(by school choir or by the entire school)* "Dear Lord and Father of Mankind."

September: Second Sunday

SUGGESTED HYMN: "Faith of Our Fathers."

STORY:

READING THE BIBLE UNDER DIFFICULTIES

It is relatively easy to picture in our imagination a time when few people had Bibles, but it is not so easy to understand that at certain times it has been a criminal offense to own a Bible.

If the early Christians had used moving-picture machines, we might have had a film showing a Christian family rising early in the morning before the neighbors were awake and making their way stealthily by devious quiet ways and hidden paths to a cave, a cellar, a secret chamber, or even an underground burying place for the dead. Here we might have watched them as they sat quietly with a few others while some one took from a hidden chest one of the sacred rolls and read to the assembled group. After reading, a few words of conference, a prayer, and possibly a hymn sung softly, these early Christians would slip back to their places in the world of activities from which they had for a few moments withdrawn themselves.

It was dangerous to read the Bible in those early days. The Roman emperors not only destroyed thousands of the valuable hand-made manuscripts of the Bible which had been produced with so much labor, but they hunted down, persecuted and even killed those who were found with any part of the Bible in their possession or who shared in the group meetings of the Christians for Bible reading. It was an exciting experience to be a Christian in those days. One thing this persecution accomplished, however. It eliminated from the Christian group all those who were not desperately in earnest about their religion.

At the time of one of these persecutions, under the Emperor Diocletian, a young Christian named Marinus was serving as an officer in the Roman army in Palestine. He had done his duty faithfully and was about to be promoted to the rank of captain. Through jealousy he was denounced by one of his fellow officers as a Christian. Marinus was at once summoned before his

superior officer and questioned. "Is it true that you are a Christian?" "Yes," replied Marinus. "Then," said the officer, "I will give you three hours to renounce your Christianity." Marinus at once went to the small Christian church at Cæsarea and told his story to the aged bishop. The bishop listened and then, taking a sword in one hand and a Bible in the other, he held them up before Marinus. "This is your choice," said he. Without hesitation Marinus grasped the Bible. He returned to his post, declared himself a Christian and, instead of promotion, received the sentence of death.

Marinus, like many another early Christian, learned that the things which are worth while always cost some one a large price.

September: Third Sunday

Suggested Hymn: "A Glory Gilds the Sacred Page.

Story:

PRINTING THE BIBLE

Few events have been of more significance in the history of the race than the invention of the art of printing. For fourteen hundred years the Christian church had written by hand every copy of the Scriptures it possessed. If you will sit down this afternoon and copy one page of your Bible and then multiply the time consumed in the process by the number of pages, you will get some idea of the labor involved in making even one copy of the Bible. It is not surprising that Bibles were scarce and too expensive to be in the possession of the ordinary Christian.

It was a great day for the Bible when the first printing press was set up by Gutenberg about the year 1450. The

first book to be printed on this first printing press of the world was the Bible. Since the appearance of that first printed Bible the printing presses have never ceased to turn out Bibles, and to-day, after more than four hundred years, the Bible is the best selling book in the world, and the printing presses can hardly keep pace with the demand.

In connection with the first printed English Bible the name of William Tyndale, known as "the father of the English Bible," should be cherished. While a young man in college Tyndale got hold of a Greek New Testament. As he read this book of which he knew so little, he was impressed with the feeling that the church had drifted far from the religion taught by Jesus. He was also seized with the desire to place a Bible in every cottage and palace in England. In a dispute with a so-called "learned man" about this time over the authority of the Pope, Tyndale said: "If God spare my life, ere many years I will cause a boy that driveth the plow to know more of the Scripture than thou dost."

Before Tyndale could carry out his project it was necessary that the Bible be translated into English, and to this task he applied himself. In spite of difficulties, opposition and persecution, the first copies of Tyndale's New Testament appeared in 1525. By this time both the king and the church were arrayed against Tyndale. Sermons were preached against him and his work, copies of the book were publicly burned.

Tyndale had been obliged to do his printing on the continent of Europe. The books were sent to England wrapped in bales of cloth and otherwise disguised. Tyndale himself was driven from one hiding place to an-

exile, bitter absence from friends, hunger and thirst and cold, great dangers and other hard and sharp fightings." At last he was betrayed by one whom he had trusted and was thrown into jail. Even here he continued his work of translation until October, 1536, when he was put to death by strangling after being condemned as a heretic. His last words were, "Lord, open the king of England's eyes!"

Surely this man deserves the title which has been given him—"The father of the English Bible." He, like Marinus, learned that the best things in life always cost some one a very large price.

September: Fourth Sunday

Suggested Hymn: "We've a Story to Tell to the Nations."

Story:

A LONG WALK FOR THE BOOK

If you had been in the frontier trading post known as Saint Louis some eighty-five years ago, you might have been startled one morning to see five strange, swarthy figures approaching from the west. If you had followed them to General Clark's headquarters in the Barracks and listened while they made known their identity and the purpose of their visit, you would have learned that they were Nez Perce Indians and that they had walked from the far Northwest, two thousand miles away, to secure "the white man's Book of Heaven." It was thus that they designated the Bible, of which they had heard from the traders and for which they had waited many years in their far Western home. They had been as-

sured that missionaries would come to them with the Book but year after year passed, the old men were dying, and still the missionaries did not come. It was then that a council of the tribes was called, and four warriors chosen to take the long journey and bring back the much-coveted Book.

General Clark received these dignified messengers kindly, and for several months they were accorded the finest hospitality which a frontier town afforded, but the Book was not to be secured. The long journey, unaccustomed luxuries, and the tragedy of disappointment, each played its part, and two of the Indians died. At last the other two resolved to undertake the long return journey. Before they started, one of the Indians made a speech which brought tears to the eyes of many. He said:

"I came to you over a trail of many moons from the setting sun. You were the friends of my fathers, who have all gone the long way. I came with one eye partly open. I go back with both eyes closed. How can I go back blind to my blind people? I made my way to you with strong arms through many enemies and strange lands that I might carry back much to them. I go back with both arms broken and empty. Two fathers came with us. They were braves of many snows and wars. We leave them asleep here by your great water and tepees. They were tired in many moons, and their moccasins wore out. My people sent me to get the white man's Book of Heaven. You took me to where you allow your women to dance as we do not ours, and the Book was not there. You took me to where they worship the Great Spirit with candles, and the Book was not there. I am going back the long trail to my people in the dark

land. You make my feet heavy with gifts, and my moccasins will wear out in carrying them, yet the Book is not among them. When I tell my poor blind people, after one more snow, in the big council, that I did not bring the Book, no word will be spoken by our old men or by our young braves. One by one they will rise up and go out in silence. My people will die in darkness and they will go on a long path to other hunting grounds. No white man will go with them, and no white man's Book to show them the way. I have no more words."

This speech made such an impression that it resulted in the going of Marcus Whitman and Jason Lee as missionaries to the great Northwest. To-day the Nez Perce Indians are Christians and they are sending of their own men and money to help carry to others the "white man's Book of Heaven" for which they waited so long.

September: Fifth Sunday

SUGGESTED HYMN: "The Morning Light Is Breaking."
STORY:

TRANSLATING THE BIBLE

The Bible was written originally in Hebrew and Greek. To-day the entire Bible, or parts of it, are printed in more than five hundred different languages. The amount of patient labor and self-sacrifice which have been put into these many Bible translations is past calculating. Frequently from twenty to forty years have been involved in the making of a single translation, and the coöperation of many people has been required. One man worked fifteen years on a translation, only to have all his labors swallowed up by the sinking of a boat.

The mere search for words to express the ideas of

Christianity is enough to test the heroic qualities of the translators. James D. Taylor tells us that while the Zulu translation was being made, an entire week was spent upon the one word, "Glory." Oftentimes no words could be found for "sin," "love," "conscience," and other terms so familiar to the Christian. In the Chinese language no word could be found for "God." The nearest approach to it was the word "ghost." In Madagascar no word could be found for "purity," so the word "whiteness" was pressed into service. When the first missionaries went to work among the Nestorians of Persia, there were no words for "wife" or "home." In Tahiti no word could be found for "faith." At another time translators were perplexed because they could find no word for "hope."

The difficulties of Bible translation are not confined to those having to do with the expression of spiritual truths. How, for example, would you translate the names of the large number of animals mentioned in the Bible to the people of Micronesia, who never had seen a four-footed beast? How would you translate Isaiah 3:18-23, with its references to anklets, crescents, pendants, bracelets, mufflers, headtires, ankle chains, sashes, perfume-boxes, amulets, festival robes, mantles, shawls, hand-mirrors, fine linen, turbans, and veils, to the Zulus, whose wardrobe consisted of a little bead work, a blanket, and a skin apron? How would you translate the many references to frost, snow, and ice to people on a tropical island, who never had experienced a temperature as low as freezing? The following is an actual translation in the Fiji Island of Isaiah 1:18b: "Though your sins be as scarlet, they shall be as white as rain." By mistake one translator made the people throw "thorn

bushes" instead of "palm branches" in the way at the time of Jesus' entry into Jerusalem.

Surely the difficulties of Bible translation have been great, but to-day we can rejoice in the fact that most of the people of the world have at least a portion of the Bible available in their own language.

The system is then said to have $2n$ degrees of freedom. If some of the particles are constrained to move on κ given curves, or more generally if there are κ given relations between the $2n$ coordinates, only $2n-\kappa$ coordinates are necessary to fix the position of the system and there are then $2n-\kappa$ degrees of freedom. *The degrees of freedom of a system may be defined to be the number of coordinates required to fix its position.*

253. Vis viva of a rigid body. When some or all of the particles of a system are rigidly connected together a simple and useful expression for the vis viva can be found. Let $(\bar{x}, \bar{y})$ be the coordinates of the centre of gravity, ϕ the angle which a straight line fixed in the body makes with a straight line fixed in space, and M the mass. The vis viva is then

$$\Sigma mv^2 = M\left\{\left(\frac{d\bar{x}}{dt}\right)^2 + \left(\frac{d\bar{y}}{dt}\right)^2\right\} + Mk^2\left(\frac{d\phi}{dt}\right)^2,$$

where Mk^2 is the constant called the moment of inertia of the body about the centre of gravity, see Art. 241.

To prove this, let $x = \bar{x} + \xi$, $y = \bar{y} + \eta$ be the coordinates of any particle m, then

$$\Sigma m\left(\frac{dx}{dt}\right)^2 = (\Sigma m)\left(\frac{d\bar{x}}{dt}\right)^2 + 2\left(\Sigma m\frac{d\xi}{dt}\right)\frac{d\bar{x}}{dt} + \Sigma\left\{m\left(\frac{d\xi}{dt}\right)^2\right\}.$$

Since $\Sigma m\xi = 0$ as in Art. 240 the middle term is zero. Hence

$$\Sigma mv^2 = M\left\{\left(\frac{d\bar{x}}{dt}\right)^2 + \left(\frac{d\bar{y}}{dt}\right)^2\right\} + \Sigma m\left\{\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2\right\}.$$

This equation expresses the proposition that *the whole vis viva of a moving system, whether rigid or not, is equal to that of a particle of mass M moving with the centre of gravity together with the vis viva of the motion relative to the centre of gravity.*

To introduce the condition that the system is rigid we change to polar coordinates by writing

$$(d\xi)^2 + (d\eta)^2 = (ds)^2 = (d\rho)^2 + (\rho d\theta)^2.$$

Remembering that $d\theta/dt$ is now the same for all the particles and equal to $d\phi/dt$ (Art. 240) and that $d\rho/dt$ is zero, we find

$$\Sigma m\left\{\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2\right\} = (\Sigma m\rho^2)\left(\frac{d\theta}{dt}\right)^2 = Mk^2\left(\frac{d\phi}{dt}\right)^2.$$

254. Examples. *Ex.* 1. An endless light string of length $2l$, on which are threaded beads of masses M and m, passes over two small smooth pegs A and B in the same horizontal line and at a distance apart a, one bead lying in each of the festoons into which the string is divided by the pegs. The lighter bead m is raised to the mid-point of AB and then let go. Show that the beads will just meet if $\dfrac{M+m}{M}=2\left(\dfrac{l}{l+a}\right)^{\frac{1}{2}}$. [Math. Tripos, 1897.]

We notice that only two positions of the system are contemplated in the problem, viz. (1) the initial position in which the bead m lies in AB, and (2) the position in which the beads are in contact. In both these cases the kinetic energy is zero. The principle of vis viva asserts that *the change of kinetic energy is equal to the work.* It immediately follows that the work done when the system passes from the first to the second position is zero. Let x be the depth below AB at which the beads meet. Then omitting the tension, Art. 248, we have

$$mgx+M\{x-\sqrt{(l^2-al)}\}=0.$$

We also have by geometry $4x^2+a^2=l^2$. Eliminating x we obtain the result.

The circumstances of the motion when the beads m, M are at any depths y, η below AB may also be deduced from the principle. We have

$$\tfrac{1}{2}(mv^2+Mv'^2)=mgy+Mg\{\eta-\sqrt{(l^2-al)}\} \quad\text{.....................}\ (1).$$

Since the sum of lengths joining m and M to A is l, we have the geometrical equation

$$\sqrt{(\tfrac{1}{4}a^2+y^2)}+\sqrt{(\tfrac{1}{4}a^2+\eta^2)}=l \quad\text{.......................}\ (2).$$

Differentiating the second equation, we have

$$\frac{yv}{\sqrt{(a^2+4y^2)}}+\frac{\eta v'}{\sqrt{(a^2+4\eta^2)}}=0 \quad\text{.............................}\ (3).$$

Joining this to (1) we have the values of v, v' when y and η have any values not inconsistent with (2).

Ex. 2. A particle of mass m has attached to it two equal weights by means of strings passing over pulleys in the same horizontal line and is initially at rest half way between them. Prove that if the distance between the pulleys be $2a$, the velocity of m will be zero when it has fallen through a space $\dfrac{4mm'a}{4m'^2-m^2}$.

[Coll. Exam.]

Ex. 3. Two pails of weights W, w, are suspended at the ends of a rope which is coiled round the perfectly rough rim of a uniform circular disc of radius a supported in a vertical plane on a smooth horizontal axis, and the pails can descend into a well so that when one comes up the other goes down. If the pails be allowed to move freely under gravity, and, when the heavier has descended a distance b from rest, a drop of water be thrown off from the highest point of the rim of the disc, prove that this drop will strike the ground at a horizontal distance x from the axis of the disc given by

$$x^2(\tfrac{1}{2}W'+W+w)=4hb\,(W-w),$$

where W' is the weight of the disc, and h is the vertical distance above the ground of the highest point of the rim of the disc. [Math. Tripos, 1897.]

The equation of vis viva gives

$$M'k^2\omega^2+(M+m)\,v^2=2\,(M-m)\,gb.$$

The theory of parabolic motion gives $x=vt$, and $h=\frac{1}{2}gt^2$. Putting $\omega=v/a$ and $k^2=\frac{1}{2}a^2$, we obtain the required value of x.

Ex. 4. Two small holes A, B are made in a smooth horizontal table, the distance apart being $2a$. A particle of mass M rests on the table midway between A and B; and a particle of mass m hangs beneath the table, suspended from M by two equal weightless and inextensible strings, passing through the two holes. The length of each string is $a(1+\sec\alpha)$. A blow J is applied to M in a direction perpendicular to AB; show that if $J^2>2Mmag\tan\alpha$, M will oscillate to and fro through a distance $2a\tan\alpha$. But if J^2 is less than this quantity and equal to $2Mmag(\tan\alpha-\tan\beta)$, the distance through which M oscillates will be

$$2a\{p(p+2)\}^{\frac{1}{2}}, \text{ where } p=\sec\alpha-\sec\beta.$$ [Coll. Ex. 1895.]

The effect of the blow J is to communicate an initial velocity $V=J/M$ to the mass M, leaving m initially at rest.

Ex. 5. Two particles M, m are connected by a string passing over a smooth pulley, the lesser mass m hangs vertically, and M rests on a plane inclined at an angle α to the vertical. M starts without initial velocity from the point of the inclined plane vertically under the pulley. Prove that M will oscillate through a distance $\dfrac{2m(M-m)h\cos\alpha}{m^2-M^2\cos^2\alpha}$ where h is the height of the pulley above the initial position of M, m is greater than $M\cos\alpha$ but less than M. [Coll. Ex. 1897.]

Ex. 6. Two equal particles connected by a string are placed in a circular tube. In the circumference is a centre of force varying as the inverse distance. One particle is initially at rest at its greatest distance from the centre of force, prove that if v, v' be the velocities with which they pass through a point 90° from the centre of force, $e^{-v^2/\mu}+e^{-v'^2/\mu}=1$. [Coll. Exam.]

Ex. 7. A thin spherical shell of mass M is driven out symmetrically by an internal explosion. Prove that if when the shell has a radius a the outward velocity of each particle be V, the fragments can never be collected by their mutual attraction unless $V^2<M/a$. [Coll. Exam.]

The attraction of a thin spherical shell on an element of itself is the same as if *half* the mass of the shell were collected at the centre.

Ex. 8. Three equal and similar particles repelling each other with forces varying as the distance are connected by equal inextensible strings and are at rest; if one string be cut, the subsequent angular velocity of either of the other strings will vary as $\sqrt{\dfrac{1-2\cos\theta}{2+\cos\theta}}$, θ being the angle between them. [Christ's Coll.]

Ex. 9. An elastic string of mass m and modulus E rests unstretched in the form of a circle of radius a. It is now acted on by a repulsive force situated in its centre whose magnitude is $\mu\,(\text{distance})^{-2}$. Prove that the radius of the circle when it next comes to rest is a root of the quadratic $r^2-ar=m\mu/E\pi$. [Coll. Exam.]

Ex. 10. A circular hoop of radius b, without mass, has a heavy particle rigidly attached to it at a point distant c from its centre, and its inner surface is constrained to roll on the outer surface of a fixed circle of radius a (b being greater than a), under the action of a repelling force from the centre of the fixed circle equal to μ times the distance. Prove that the period of small oscillations of the hoop will be $2\pi\dfrac{b+c}{a}\left(\dfrac{b-a}{c\mu}\right)^{\frac{1}{2}}$.

Prove that when $c=b$, all oscillations large or small have the same period; and prove further that in the general case the hoop may be started so that it will continue to roll with uniform angular velocity equal to $\{\mu b/(b-a)\}^{\frac{1}{2}}$.

[Math. Tripos, 1886.]

The following is a simple (but not necessarily the shortest) method of writing down the equation of vis viva in problems of this kind. Having selected some independent variable to fix the position of the system, say, the inclination θ of the straight line joining the centres C, O of the two circles to the vertical, we find the coordinates x, y of the particle in terms of θ by projecting OC, CP on the vertical and horizontal. The vis viva, being the sum of $m(dx/dt)^2$ and $m(dy/dt)^2$, follows immediately. Equating the half of this sum to the force function $\frac{1}{2}m\mu \,.\, CO^2 + C$ we have an equation giving $d\theta/dt$ in terms of θ.

It is then easily seen that, if the constant C be properly chosen, the value of $d\theta/dt$ reduces to the constant given in the question. To find the small oscillations, we differentiate the equation of vis viva and reject the squares of θ.

When $c=a$, the path of the particle is an epicycloid and the oscillations large or small are, by Art. 211, tautochronous.

255. Rotating field of force. When a particle moves in a field of force which rotates round the origin O with a uniform angular velocity n, an integral of the equations of motion can be found which reduces to that of vis viva when $n=0$.

Let $O\xi$, $O\eta$ be two rectangular axes which rotate with the field of force, and let X, Y be the component accelerating forces. We then have by Art. 227

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - 2n\frac{d\eta}{dt} - n^2\xi &= X \\ \frac{d^2\eta}{dt^2} + 2n\frac{d\xi}{dt} - n^2\eta &= Y \end{aligned}\right\} \quad \text{..................(1).}$$

Multiplying these by $d\xi/dt$ and $d\eta/dt$ and adding, we find

$$\frac{d\xi}{dt}\frac{d^2\xi}{dt^2} + \frac{d\eta}{dt}\frac{d^2\eta}{dt^2} - n^2\left(\xi\frac{d\xi}{dt} + \eta\frac{d\eta}{dt}\right) = X\frac{d\xi}{dt} + Y\frac{d\eta}{dt},$$

$$\therefore\ \frac{1}{2}\left\{\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2\right\} - \frac{n^2}{2}(\xi^2+\eta^2) = \int(Xd\xi + Yd\eta) \text{......(2).}$$

We introduce the condition that the field of force rotates by making X, Y such functions of ξ, η only that $X = dU/d\xi$ and $Y = dU/d\eta$. Then U is a function of ξ, η only and not of t. The equation then becomes

$$\tfrac{1}{2}(v^2 - n^2r^2) = U + C \text{........................(3),}$$

where v is the velocity of the particle relatively to the moving axes and r is the radius vector.

We may notice that if U be expressed in terms of the coordinates x, y referred to fixed axes, the expression will contain t also, except when the force is central and tends to O.

The equation, when written in the form (2), is a slight extension of that given by Jacobi in the *Comptes Rendus*, Tome III. p. 59, 1836.

If V be the space velocity of the particle, A the angular momentum about O referred to a unit of mass, then

$$V^2 - 2nA = v^2 - r^2n^2 \quad\text{.....................(4).}$$

The equation of Jacobi then becomes

$$\tfrac{1}{2}V^2 - nA = U + C \quad\text{.......................(5).}$$

To prove the relation (4), let p be the perpendicular from O on the tangent to the relative path. Since V is the resultant of v and nr, (the latter being perpendicular to r), we have

$$V^2 = v^2 + n^2r^2 + 2v\,.\,np, \quad A = vp + nr^2,$$

the second equation being obtained by taking moments about O. The equation (4) follows at once.

An example of a rotating field of force is met with in astronomy. If the components of a binary star describe circles about their common centre of gravity, the force is always the same at the same point of the rotating plane. Jacobi's integral will therefore apply to the motion of a satellite moving in that plane, provided it is of such insignificant mass that the motions of the primaries are undisturbed by its attraction.

256. When the particle moves in space of two dimensions and the field of force rotates about a perpendicular axis with a variable angular velocity ϕ' we may obtain an extension of the equations.

We know that $\frac{1}{2}dV^2/dt$ is equal to the sum of the virtual moments of the forces divided by dt, (Art. 246), hence

$$\begin{aligned}\tfrac{1}{2}dV^2/dt &= Xu + Yv\\ &= X\xi' + Y\eta' + \phi'(\xi Y - \eta X).\end{aligned}$$

But $dA/dt = \xi Y - \eta X$ by taking moments about the origin, hence

$$\frac{1}{2}\frac{dV^2}{dt} - \phi'\,.\,\frac{dA}{dt} = \frac{dU}{dt} \quad\text{................................ (6),}$$

where U is a function of the moving coordinates ξ, η, z. When ϕ' is constant, this can be integrated and we obtain the equation (5).

When *a system of particles* moving in a given rotating field of force is under consideration, we have for each an equation similar to (6). Multiplying these by the masses of the particles and adding the products, we have an extended equation

of vis viva. If $2T$ be the vis viva, A the angular momentum of the system, U the force function, this equation is

$$T - \phi' A = U + C \qquad (7),$$

where ϕ' is the angular velocity of the field supposed to be constant. In this form we may omit from U all the actions and reactions which disappear in the principle of virtual work.

257. Coriolis' theorem on relative vis viva. A system of particles is referred to moving axes $O\xi$, $O\eta$. Supposing the system at any instant to become fixed to the moving axes, let us calculate what would *then* be the effective forces on the system. If we apply these as additional impressed forces on the system, but reversed in direction, we may use the equation of vis viva to determine the relative motion as if the axes were fixed in space.

Let m_1, m_2, &c. be the masses of the particles; (X_1, Y_1), (X_2, Y_2), &c. the components of the impressed forces. Let also p, q be the resolved velocities of the origin, then, including these as explained in Art. 227, the equations of motion of any representative particle m are

$$\left.\begin{aligned} m\left\{\frac{d^2\xi}{dt^2} - \omega^2\xi - \frac{1}{\eta}\frac{d}{dt}(\eta^2\omega) + \frac{dp}{dt} - q\omega\right\} &= X \\ m\left\{\frac{d^2\eta}{dt^2} - \omega^2\eta + \frac{1}{\xi}\frac{d}{dt}(\xi^2\omega) + \frac{dq}{dt} + p\omega\right\} &= Y \end{aligned}\right\} \qquad (1),$$

where $\omega = d\phi/dt$.

The left-hand sides of these equations measure the components of *the effective forces* on the particle m, Art. 227. The corresponding components on an imaginary particle of the same mass m attached to the moving axes and momentarily coinciding with the real particle are found by treating ξ, η as constants. These are

$$\left.\begin{aligned} \left\{-\omega^2\xi - \eta\frac{d\omega}{dt} + \frac{dp}{dt} - q\omega\right\} &= X_0 \\ \left\{-\omega^2\eta + \xi\frac{d\omega}{dt} + \frac{dq}{dt} + p\omega\right\} &= Y_0 \end{aligned}\right\} \qquad (2).$$

These we represent by X_0, Y_0 for the sake of brevity.

Transposing these terms to the other sides of the equations of motion, we have

$$\left.\begin{aligned} m\left(\frac{d^2\xi}{dt^2} - \omega\frac{d\eta}{dt}\right) &= X - X_0 \\ m\left(\frac{d^2\eta}{dt^2} + \omega\frac{d\xi}{dt}\right) &= Y - Y_0 \end{aligned}\right\} \qquad (3).$$

These equations may also be used to supply another proof of the theorem in Art. 197.

Multiplying these respectively by $d\xi/dt$, $d\eta/dt$ and adding, we have, as in Art. 255,

$$m\left\{\frac{d\xi}{dt}\frac{d^2\xi}{dt^2} + \frac{d\eta}{dt}\frac{d^2\eta}{dt^2}\right\} = (X - X_0)\frac{d\xi}{dt} + (Y - Y_0)\frac{d\eta}{dt}.$$

Summing this representative equation for all the particles and integrating

$$\tfrac{1}{2}\Sigma m\left\{\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2\right\} = \Sigma\int\{(X - X_0)\,d\xi + (Y - Y_0)\,d\eta\} \qquad (4).$$

If the axes rotate round a fixed origin with a uniform angular velocity, ω is constant and p, q are zero. The equation of Coriolis then takes the simpler form

$$\tfrac{1}{2}\Sigma mv^2 = U + \tfrac{1}{2}\omega^2\Sigma mr^2 + C \qquad (5),$$

where r is the distance of the particle m from the origin and v is its velocity *relatively to the axes.* For a single particle this is the same as Jacobi's integral.

If the angular velocity ω is not uniform and p, q not zero, the system of additional forces (X_0, Y_0) is not conservative and the integration in (4) cannot be effected except in special cases. The equation is however still important, for the first step in the integration of the equations (1) must be to eliminate the unknown reactions, if any such exist. Now the equation (4) is free from all the reactions which would disappear in the principle of vertical work, and that equation therefore supplies us at once with one result at least of the elimination.

For the purposes of this proposition the forces measured by X_0, Y_0 are called *the forces of moving space.* When the origin of coordinates is fixed, these take the simple form

$$X_0 = -\omega^2\xi - \eta\frac{d\omega}{dt}, \qquad Y_0 = -\omega^2\eta + \xi\frac{d\omega}{dt} \quad\text{.....................}\quad (6).$$

This theorem is due to Coriolis; see the *Journal Polytechnique*, 1831.

258. Laisant's theorem. *Ex.* A particle moves under the action of a force whose Cartesian components are $X = v^n\dfrac{dU}{dx}$, $Y = v^n\dfrac{dU}{dy}$, where v is the velocity. Prove that the equation of vis viva is $v^{2-n} = (2-n)\,U + C$.

See the *Bulletin de la Société Mathématique*, 1893, vol. XXI.

Moments and Resolutions.

259. The equation of Moments. If P, Q are the components of the force on a single particle resolved along and transverse to the radius vector, it is clear that Qr is equal to the moment of the forces about the origin. Representing this moment by M, the transverse polar equation of motion becomes

$$m\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = M \quad\text{.....................}\quad (1).$$

260. When a system of *mutually attracting particles* moves under the action of external forces we have by adding together the transverse polar equations of each particle

$$\Sigma m\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \Sigma M \quad\text{....................}\quad (2).$$

If R be the attraction of m_1 on m_2, the reaction of m_2 on m_1 is $-R$, and the sum of the moments of these two must disappear from the right-hand side. If then the external forces are such that their resultant passes through the origin, we have $\Sigma M = 0$, and therefore by integration

$$\Sigma mr^2\frac{d\theta}{dt} = H \quad\text{......................}\quad (3),$$

where H is a constant. This equation expresses the proposition that *when a system of mutually attracting particles moves under the action of external forces such that the sum of the moments about a fixed point is zero, the sum of the angular momenta of all the particles about that point is constant.* For example, if any number of mutually attracting planets move under the influence of a fixed sun, the sum of their angular momenta is constant. See also Art. 93.

Since $x\,dy - y\,dx = r^2 d\theta$ (Art. 7), the equation (3) of moments when written in Cartesian coordinates takes the form

$$\Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = H \qquad (4).$$

261. Rigid system. When a system of particles is rigid it is useful to have an expression for the resultant angular momentum about the origin. Let $(\bar{x}, \bar{y})$ be the coordinates of the centre of gravity, ϕ the angle a straight line fixed in the body makes with a straight line fixed in space, and M the mass. *The angular momentum of the whole mass is then*

$$H = M\left(\bar{x}\frac{d\bar{y}}{dt} - \bar{y}\frac{d\bar{x}}{dt}\right) + Mk^2\frac{d\phi}{dt},$$

where Mk^2 is the moment of inertia about the centre of gravity. See Art. 241.

To prove this, let (x, y) be the coordinates of the particle m, then $x = \bar{x} + \xi$, $y = \bar{y} + \eta$. Remembering that $\Sigma m\xi = 0$, $\Sigma m\eta = 0$ as in Art. 239, we find by substitution that

$$\Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = (\Sigma m)\left(\bar{x}\frac{d\bar{y}}{dt} - \bar{y}\frac{d\bar{x}}{dt}\right) + \Sigma m\left(\xi\frac{d\eta}{dt} - \eta\frac{d\xi}{dt}\right).$$

Since $d\bar{x}/dt$, $d\bar{y}/dt$ are the components of the velocity of the centre of gravity, the first term is the moment of the velocity of a particle of mass M placed at the centre of gravity and moving with it. The equation therefore asserts that *the angular momentum about any point is equal to that of the whole mass collected at the centre of gravity together with the angular momentum round the centre of gravity of the relative motion.*

To introduce the condition that the system is rigid we change to polar coordinates by writing $\xi d\eta - \eta d\xi = \rho^2 d\theta$. The second term then becomes $\Sigma m\rho^2\dfrac{d\theta}{dt}$. Remembering that $d\theta/dt$ is the

same for every particle and equal to $d\phi/dt$ (Art. 240), this term becomes $Mk^2 \dfrac{d\phi}{dt}$.

It follows that, when a rigid body is acted on by any forces whose moment about the origin is G, the equation of moments is

$$\frac{d}{dt}\left[M\left(\bar{x}\frac{d\bar{y}}{dt} - \bar{y}\frac{d\bar{x}}{dt}\right) + Mk^2\frac{d\phi}{dt}\right] = G.$$

262. *Ex.* 1. A particle moves in a field of force defined by the force function

$$U = mf(r) + \frac{mF(\theta)}{r^2}.$$

Show how to find the coordinates r, θ in terms of the time.

The force transverse to the radius vector is $Q = dU/rd\theta$. The equation of moments therefore becomes $\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right) = \dfrac{1}{r^2}\dfrac{dF}{d\theta}$. Multiplying by $r^2d\theta/dt$, the integration can be effected and we find

$$\left(r^2\frac{d\theta}{dt}\right)^2 = 2F(\theta) + A \quad\text{.................................}\quad (1),$$

where A is an arbitrary constant. This integral is equivalent to a result given by both Jacobi and Bertrand.

The equation of vis viva is

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = 2f(r) + \frac{2F(\theta)}{r^2} + C \quad\text{..........................}\quad (2).$$

Eliminating $d\theta/dt$ by the help of (1) we arrive at an equation giving dt/dr as a function of r. The determination of t in terms of r has thus been reduced to an integration. The relation between θ and t may then be found from (1) by another integration.

Ex. 2. A particle is placed at rest at the point $x=0$, $r=a$ in a field defined by $U = m\dfrac{a^4x}{r^3}$. Show by writing down the equations of vis viva and moments that the path is a circle.

263. The equation of resolution. If a system of particles moves under the action of external forces, we have by resolving parallel to the axis of x, (Art. 236),

$$\Sigma m\frac{d^2x}{dt^2} = \Sigma mX,$$

where X is the typical accelerating force on the particle m. In this equation we may omit the mutual attractions of the particles, for the action and reaction being equal and opposite, these disappear in the resolution.

If any direction fixed in space exist such that the sum of the components of the impressed forces in that direction is zero, we

can take the axis of x parallel to that direction. We then have

$$\Sigma mX = 0, \quad \therefore \ \Sigma m \frac{dx}{dt} = A,$$

where A is a constant. This result is the same as that already arrived at, and more fully stated, in Art. 92.

264. Summary of methods of integration. When the system of particles moves in a given field of force the equation of vis viva in general supplies one integral of the equations of motion. If the system has only one degree of freedom, this integral is sufficient to determine the motion.

When another integral is required, there is no general method of proceeding. We usually search if there is any direction fixed in space in which the sum of the resolved parts of the forces is zero, or any fixed point about which the sum of the moments is zero. In either of these cases an additional integral is supplied by the methods of Arts. 263 and 260. The first case usually occurs when the acting force is gravity, the second when the force is central.

When these methods fail we have recourse to some artifice suited to the problem. Suppose that we have some reason for believing that a particle describes a certain path, we constrain the particle by a smooth curve. If the pressure can be made zero by the proper initial conditions, the constraint may be removed and the particle will describe the path freely, Art. 193.

265. Examples. *Ex.* 1. Two particles, of masses m, M, placed on a smooth table, are connected by a string of length $a+b$, which passes through a fine ring fixed at a point O on the table. The particles are projected with velocities U and V perpendicularly to the portions of the string attached to them, and the initial lengths are respectively a and b. Find the motion.

Let (r, θ), (ρ, ϕ) be the polar coordinates of m and M at the time t. By the principles of angular momentum and vis viva, we have

$$r^2 \frac{d\theta}{dt} = Ua, \qquad \rho^2 \frac{d\phi}{dt} = Vb \quad \text{.............................} \quad (1),$$

$$m\left\{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right\} + M\left\{\left(\frac{d\rho}{dt}\right)^2 + \rho^2\left(\frac{d\phi}{dt}\right)^2\right\} = mU^2 + MV^2 \text{.........} \ (2).$$

We have also the geometrical equation

$$r + \rho = a + b \quad \text{......................................} \quad (3).$$

Eliminating ρ, θ, ϕ, we find

$$(M+m)\left(\frac{dr}{dt}\right)^2 + \frac{mU^2a^2}{r^2} + \frac{MV^2b^2}{(a+b-r)^2} = mU^2 + MV^2 \text{..............} (4).$$

In this differential equation, the variables can be separated and thus t can be expressed in terms of r by an integral. The integration cannot be generally effected.

If the system oscillate, the extreme positions are determined by putting $dr/dt=0$. We thus have

$$\frac{mU^2a^2}{r^2}+\frac{MV^2b^2}{(a+b-r)^2}-(mU^2+MV^2)=0 \quad\text{..................... (5).}$$

Since the left-hand side is positive when $r=0$ and $r=a+b$ and vanishes when $r=a$ there is a second positive root less than $a+b$. This second root may be proved to be greater or less than a according as mU^2/a is greater or less than MV^2/b. These values of r determine the extreme positions of the system. We notice that if V be very small, the second root is very nearly equal to $a+b$.

If $V=0$ the particle M arrives at the origin, but the appearance when $r=a+b$ of the singular form 0/0 in the equation (5) is a warning that the motion changes its character in this case. In fact if the third term on the left-hand side of (4) is removed, the velocity of arrival at O is finite instead of being infinitely great.

To find the tension T of the string, we use the radial equation of motion for one of the particles. This gives

$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=-\frac{T}{m}.$$

Differentiating (4) we find dr/dt in terms of r and after some slight reductions

$$T=\frac{Mm}{M+m}\left(\frac{U^2a^2}{r^3}+\frac{V^2b^2}{(a+b-r)^3}\right).$$

The string therefore does not become slack.

Ex. 2. Two particles whose masses are in the ratio 1 : 2 lie on a smooth horizontal table, and are connected by a string that passes through a small ring in the table: the string is stretched and the particles are equidistant from the ring: the lighter particle is then projected at right angles to its portion of the string. Prove that the other particle will strike the ring with half the initial velocity of the first particle. [Coll. Ex. 1896.]

Ex. 3. One A of two particles of equal mass, without weight, and connected by an inelastic string moves in a straight groove. The other B is projected parallel to the groove, the string being stretched. Prove that the greatest tension is four times the least. [Coll. Ex.]

Reduce A to rest, then B is acted on by T and $T\cos\theta$, the latter being parallel to the groove, where θ is the angle AB makes with the groove. The particle B now describes a circle, and the normal and tangential resolutions give the angular velocity and the tension.

Ex. 4. Two particles m, M, are connected by a string, of length $a+b$, which passes through a hole in a smooth table; M hangs vertically at a depth b below the hole, m is projected horizontally and perpendicularly to the string with velocity V from a point on the table distant a from the hole. Prove that if M just rise to the table, $mV^2(2ab+b^2)=2Mgb(a+b)^2$. Prove also that if M oscillates,

$$mV^2+2Mga>3(M^2mV^2g^2a^2)^{\frac{1}{3}}.$$

What is the motion if $mV^2=Mga$?

Ex. 5. Two *small* spheres of masses m and $2m$ are fixed at the ends of a weightless rigid rod AB which is free to turn about its middle point O; the heavier sphere rests on a horizontal table, the rod making an angle 30° with it. If a sphere of mass m falling vertically with velocity u strike the lighter sphere directly, prove that the impulse which the heavier sphere ultimately gives to the table is $\frac{4}{5}mu(1+e)$, where e is the coefficient of restitution between the two spheres, the table being perfectly inelastic. [Coll. Ex. 1893.]

At the first impact we take moments for the two particles m, $2m$ about O to avoid the reaction at O. We therefore have $3mv'a = Ra\cos\alpha$, $m(u'-u) = -R$ where $\alpha = 30°$. At the moment of greatest compression the velocity of approach of the centres is zero, $\therefore u' = v'\cos\alpha$, and $R = \frac{4}{5}mu$. Since the complete value of R is found by multiplying this by $1+e$, the velocity of either end of the rod after impact is $\frac{4}{15}u\cos\alpha(1+e)$. The balls m and $2m$ rotate with the rod round O through some angle, and $2m$ finally hits the table with a velocity v'. Taking the same equation of moments as before $R'a\cos\alpha = 3mv'a$, $\therefore R' = \frac{4}{5}mu(1+e)$.

Ex. 6. One end of a string of length l is attached to a small ring of mass m which can slide freely on a smooth horizontal wire, and the other end supports a heavy particle of mass m'. If this particle be held displaced in the vertical plane containing the groove, the string being straight and then let go, prove that the path of m' is part of an ellipse whose semi-axes are l, $lm/(m+m')$, the major axis being vertical. [Coll. Ex. 1896.]

Only the horizontal resolution and the geometrical equation are required.

Ex. 7. A rectangular block of wood of mass M is free to slide between two smooth horizontal planes, and in it is inserted a smooth tube in the shape of a quadrant of a circle of radius a, one of the bounding radii lying along the lower plane, and the other being vertical. A particle of mass m is shot into the tube horizontally with velocity V, rebounds from the lower plane, and leaves the tube again with a relative velocity V', prove that

$$V'^2 = e^2V^2 - 2ga(1-e^2)(M+m)/M,$$

where e is the coefficient of restitution for the lower plane. [Coll. Ex. 1895.]

Ex. 8. If in the case of three equal particles the units are so chosen that the energy integral is $\frac{1}{2}(v_1^2+v_2^2+v_3^2) = \frac{1}{r_{23}} + \frac{1}{r_{31}} + \frac{1}{r_{12}} - \frac{1}{r}$, where r_{12} is the distance between the particles whose velocities are v_1 and v_2, and if r is a positive constant, the greatest possible value of the angular momentum of the system about its centre of inertia is $\frac{3}{2}\sqrt{(2r)}$. [Math. Tripos, 1893.]

Ex. 9. Two equal particles are initially at rest in two smooth tubes at right angles to each other. Prove that whatever be their positions and whatever their law of attraction, they will reach the intersection of the tubes together.

[Coll. Ex.]

Ex. 10. Three mutually attracting particles, of masses m_1, m_2, m_3, are placed at rest within three fixed smooth tubes Ox, Oy, Oz at right angles to each other. The attraction between any two, say m_1, m_2, is $\mu m_1 m_2 r_3^{\kappa}$ where r_3 is the distance. If the triangle joining the particles always remains similar to its initial form, prove that the initial distances satisfy the equations

$$\frac{m_2m_3r_1^{\kappa-1}}{m_2+m_3-m_1} = \frac{m_3m_1r_2^{\kappa-1}}{m_3+m_1-m_2} = \frac{m_1m_2r_3^{\kappa-1}}{m_1+m_2-m_3}.$$

266. Double answers. *Ex.* A cube, of mass M, constrained to slide on a smooth horizontal table, has a fine tube ACB cut through it in the vertical plane through its centre of gravity, the extremities A, B being on the same horizontal line and the tangents at A, B horizontal. A particle, of mass m, is projected into the tube at A with velocity V, deduce analytically from the equations of linear momentum and vis viva that the velocity of emergence at B is also V.

Let u, v be the velocities of the cube and particle at emergence. The principles referred to give

$$Mu+mv=mV, \qquad Mu^2+mv^2=mV^2.$$

These give two solutions, viz. (1) $u=0$, $v=V$, and (2) $u=2mV/S$, $v=(m-M)V/S$, where $S=m+M$. To interpret these we notice that there are two sets of initial conditions which give the same linear momentum and vis viva. These are determined by the values of u, v just written down. We have therefore really solved two problems and have thus obtained two results.

To distinguish the solutions, we investigate the intermediate motion. Let P be any point in the tube and let p be the tangent of the angle the tangent makes with the horizon. If u, v now represent the horizontal velocities at P, the same two principles give

$$Mu+mv=mV, \qquad Mu^2+m(v^2+p^2x'^2)=mV^2,$$

where $x'=v-u$ is the relative velocity. These give

$$v=\frac{V}{M+m}\left\{m \pm M\left(1+p^2\frac{M+m}{M}\right)^{-\frac{1}{2}}\right\}.$$

Now $v=V$ initially when $p=0$, hence the radical must have the positive sign and must keep that sign until it vanishes. On emergence therefore, when p is again zero, $v=V$. The negative sign of the radical evidently gives the initial conditions of the other problem.

267. Bodies without mass. *Ex.* 1. A heavy bead is free to slide along a rod whose ends move without friction on a horizontal circle; prove that when the mass of the rod is negligible compared with that of the bead, the bead will, when started, continue to slide along the rod with an acceleration varying inversely as the cube of its distance from the middle point. [Math. Tripos, 1887.]

The reaction between the rod and the particle is zero because the rod has no mass. To prove this, let R be the reaction, M the mass of the rod, then, taking moments about the centre O of the circle, we have $M\kappa^2 d\omega/dt=Rp$, where ω is the angular velocity of the rod. Hence $R=0$ when $M=0$.

The particle P, being not acted on by any horizontal force, describes a straight line in space with uniform velocity b. If x be the distance of P from the middle point C of the rod; a, c, the perpendiculars from O on the path and on the rod, we have $x^2+c^3=OP^2=a^2+b^2t^2$.

This gives $$d^2x/dt^2=b^2(a^2-c^2)/x^3.$$

Ex. 2. A rigid wire without mass is formed into an arc of an equiangular spiral and carries a heavy particle fixed in the pole. If the convexity of the wire be placed in contact with a perfectly rough horizontal plane prove that the point of contact will move with a uniform acceleration equal to $g\cot\alpha$, where α is the angle of the spiral. [Math. Tripos, 1860.]

268. Equation of the path. Let P, Q be the resolved accelerating forces acting on the particle respectively along and perpendicular to the radius vector. Let P be regarded as *positive when acting towards the origin.* The equations of motion are

$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=-P, \qquad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=Q \quad \ldots\ldots\ldots\ldots\ldots(1).$$

To find the path we eliminate t. The second equation, after multiplication by $r^3d\theta/dt$ and integration, as in Art. 262, becomes

$$\left(r^2\frac{d\theta}{dt}\right)^2=h^2+2\int Qr^3d\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

For the sake of brevity we represent the right-hand side by H^2. Putting also $u=1/r$, we find $d\theta/dt=Hu^2$. We then have

$$\frac{dr}{dt}=-\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}=-H\frac{du}{d\theta},$$

$$\therefore \frac{d^2r}{dt^2}=-\frac{d}{d\theta}\left(H\frac{du}{d\theta}\right).Hu^2.$$

Substituting in the first equation of motion

$$Hu^2\frac{d}{d\theta}\left(H\frac{du}{d\theta}\right)+H^2u^3=P,$$

$$\therefore H^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)+\tfrac{1}{2}u^2\frac{du}{d\theta}\frac{dH^2}{d\theta}=P.$$

Replacing H^2 by its value given in (2),

$$\left(\frac{d^2u}{d\theta^2}+u\right)\left(h^2+2\int\frac{Q}{u^3}d\theta\right)+\frac{Q}{u^3}\frac{du}{d\theta}=\frac{P}{u^2} \ldots\ldots\ldots\ldots\ldots(3).$$

This is *Laplace's differential equation of the path of the particle.* The forces P, Q being given in terms of the coordinates u, θ, of the moving particle, this equation, when solved, will determine u as a function of θ, and thus lead to the equation of the path. To find the motion along the path we use equation (2). Substituting in that equation the value of u in terms of θ we find by integration the time t at which the particle occupies any given position.

The polar differential equation of the path cannot be integrated except for special forms of the forces P, Q. If $Q=0$, the equation takes the form

$$\frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

This can be integrated when P is a function of u alone, a case which is considered in the chapter on central forces. It can also be integrated when $P=u^2F(\theta)$, the method of solution being that shown in Art. 122.

When $P=u^3F(\theta)$ the equation is linear. If one solution of the differential equation is known, say $u=\phi(\theta)$, the general integral may be determined by substituting $u=z\phi(\theta)$. After integration we find $z=A+B\int[\phi(\theta)]^{-2}d\theta$.

269. When $P=u^3F(\theta)$, $Q=u^3f'(\theta)$, *the differential equation of the path takes the linear form*

$$\left(\frac{d^2u}{d\theta^2}+u\right)\{h^2+2f(\theta)\}+f'(\theta)\frac{du}{d\theta}-F(\theta)u=0 \ldots\ldots\ldots\ldots(5).$$

The various cases in which this equation can be integrated are enumerated in treatises on Differential Equations.

By multiplying the equation by the proper factor we can make the left-hand side a perfect differential. Conversely choosing any factor, we can find the relation between P and Q that this may be the proper integrating factor. If we wish *the relation between P, Q to be independent of the initial conditions*, the terms containing h^2 as a factor must be made a perfect differential independently of the remaining terms. The coefficient of h^2 is $\frac{d^2u}{d\theta^2}+u$ and this is made a perfect differential by either of the factors $\sin\theta$ or $\cos\theta$. The remaining terms must therefore also become a perfect differential by the same factor. The condition that $L\frac{d^2u}{d\theta^2}+M\frac{du}{d\theta}+Nu$ is a perfect differential is $N-\frac{dM}{d\theta}+\frac{d^2L}{d\theta^2}=0$, and the integral is known to be $L\frac{du}{d\theta}+\left(M-\frac{dL}{d\theta}\right)u$.

Multiplying equation (5) by $\sin\theta$, the product is a perfect differential if

$$\{2f(\theta)-F(\theta)\}\sin\theta-\frac{d}{d\theta}\{\sin\theta f'(\theta)\}+2\frac{d^2}{d\theta^2}\{\sin\theta f(\theta)\}=0,$$

which reduces at once to $$\frac{P}{u^3}=\frac{d}{d\theta}\frac{Q}{u^3}+3\cot\theta\frac{Q}{u^3} \quad\text{.................. ....(6).}$$

The integral, since $f'(\theta)=Q/u^3$, becomes

$$\left(h^2+2\int\frac{Q}{u^3}d\theta\right)\left(\sin\theta\frac{du}{d\theta}-\cos\theta u\right)-\frac{Q}{u^3}\sin\theta u=C\quad\text{... ....(7),}$$

where C is a constant. This is a linear equation of the first order and can be integrated a second time when Q/u^3 is given as a function of θ. *The determination of the path can therefore be reduced to integration when the relation* (6) *is satisfied.*

In the same way, if we multiply (5) by $\cos\theta$, we find that the product is a perfect differential if $$\frac{P}{u^3}=\frac{d}{d\theta}\frac{Q}{u^3}-3\tan\theta\frac{Q}{u^3}\quad\text{..............................(8),}$$

and the integral is $$\left(h^2+2\int\frac{Q}{u^3}d\theta\right)\left(\cos\theta\frac{du}{d\theta}+\sin\theta u\right)-\frac{Q}{u^3}\cos\theta u=C'\quad\text{......(9),}$$
which is linear and can be integrated a second time.

Another case in which the integration of (3) can be effected may be deduced from Art. 262. The equation (3) is

$$\frac{d}{d\theta}\left[H^2\left\{\left(\frac{du}{d\theta}\right)^2+u^2\right\}\right]=\frac{P}{u^3}\frac{2u\,du}{d\theta}+\frac{2Q}{u^3}u^2.$$

If then $\frac{P}{u^3}=f(u)+2\int\frac{Q}{u^3}d\theta$, the integral is

$$\left\{h^2+2\int\frac{Q}{u^3}d\theta\right\}\left\{\left(\frac{du}{d\theta}\right)^2+u^2\right\}=2\int f(u)\,u\,du+2u^2\int\frac{Q}{u^3}d\theta+C\text{.........(10).}$$

270. *Ex.* 1. If $P=u^3F(\theta)$ and $Q=P\tan\theta$, prove that $u=A\sin\theta$ is a particular solution of the linear equation (5). Thence obtain the general integral by putting $u=z\sin\theta$, where z is a function of θ which is determined by solving a linear equation of the first order.

Ex. 2. A particle moves under the forces

$$P=\mu u^3(3+5\cos 2\theta),\qquad Q=\mu u^3\sin 2\theta;$$

prove that an integral of its motion is

$$h^2\left\{\frac{du}{d\theta}\sin\theta-u\cos\theta\right\}+\mu\left\{\tfrac{1}{2}(\sin\theta-\sin 3\theta)\frac{du}{d\theta}+\cos 3\theta u\right\}=C.$$

Obtain also a similar integral if

$$P=\mu u^3 \cos n\theta \left\{n+\frac{3 \tan n\theta}{\tan \theta}\right\}, \qquad Q=\mu u^3 \sin n\theta.$$

[Coll. Exam. 1892.]

Ex. 3. If the Cartesian accelerating forces X, Y are unrestricted, prove that the differential equation of the path is

$$(A+2\int X dx)\frac{d^2y}{dx^2}+X\frac{dy}{dx}-Y=0,$$

where A is a constant depending on the initial conditions.

Prove also that the determination of y as a function of x can be reduced to integration when both X, Y are functions of x only.

Ex. 4. If X and Y/y are functions of x only, the differential equation of the path is linear. Prove that it can be integrated when $Y=y\dfrac{dX}{dx}$, and that the first integral is

$$(A+2\int X dx)\frac{dy}{dx}-Xy=C.$$

Prove also that when $\dfrac{Y}{y}=\dfrac{dX}{dx}+\dfrac{3X}{x}$, the differential equation can be integrated and that the first integral is

$$(A+2\int X dx)\,x\frac{dy}{dx}-(A+2\int X dx+xX)\,y=C.$$

Ex. 5. Prove that the Cartesian equations of motion can be completely integrated when the force function satisfies

$$\frac{d^2U}{dx^2}-\frac{d^2U}{dy^2}=\kappa\frac{d^2U}{dxdy}.$$

To prove this we notice that $\qquad U=\phi(y+ax)+\psi(y+a'x),$

where a, a' are the roots of $a^2-\kappa a=1$. We then change the variables to $\xi=y+ax$ and $\eta=y+a'x$. The new coordinates ξ, η are also rectangular. The equations of motion become $d^2\xi/dt^2=\phi'(\xi)$, $d^2\eta/dt^2=\psi'(\eta)$, which may be solved as in Art. 122.

Ex. 6. If the direction of the acting force is always a tangent to the direction of motion, as in the case of a resisting medium, prove that the path is a straight line. Consider the resolution along the normal.

Ex. 7. If the direction of the force is always perpendicular to the path, prove that the velocity is constant.

Superposition of Motions.

271. A particle is constrained to describe a fixed curve. When projected from a point A with a velocity u_1 under the action of any forces the velocity and pressure at any point P are v_1 and R_1. When projected with a velocity u_2 from the same point A under a second system of forces the velocity and pressure at P are v_2

and R_2. When the particle is projected from A with a velocity u such that $u^2=u_1^2+u_2^2$, and moves under the action of both systems of forces, the velocity and pressure at P are v and R. It is required to prove that

$$v^2=v_1^2+v_2^2, \quad R=R_1+R_2.$$

To prove this we write down the two equations for each of the three types of motion. Representing for the sake of brevity the normal components of accelerating force by N_1, N_2, N_1+N_2, we have

$$\begin{aligned} v_1^2-u_1^2 &= 2\int(X_1dx+Y_1dy), & v_1^2/\rho &= N_1+R_1/m, \\ v_2^2-u_2^2 &= 2\int(X_2dx+Y_2dy), & v_2^2/\rho &= N_2+R_2/m, \\ v^2-u^2 &= 2\int\{(X_1+X_2)\,dx+(Y_1+Y_2)\,dy\}, & v^2/\rho &= N_1+N_2+R/m, \end{aligned}$$

the limits of integration being always from the point A to P.

The results follow at once by subtracting from the third equation the sum of the other two.

272. The following corollary will be found useful.

A particle can describe a curve freely under the action of certain forces, the velocity at some point A being u_1. If the particle is now constrained to describe the same curve the velocity at A being changed to u_2, then the pressure at any point P is C/ρ, where ρ is the radius of curvature at P, and C is the constant $m(u_2^2-u_1^2)$.

To prove this we notice that when the velocity at A is u_1 and the forces act on the particle, the pressure is $R_1=0$. If the velocity at A were u' and no forces acted on the particle, the pressure at P would be mu'^2/ρ. Superimposing these two states and putting $u'^2=u_2^2-u_1^2$, the theorem follows at once.

273. We may also deduce the following theorem due to Ossian Bonnet. If a particle can freely describe the same curve under two different systems of forces, the velocities at some point A being respectively u_1 and u_2, then the particle can describe the same path under both systems of forces provided the velocity at A is u, where $u^2=u_1^2+u_2^2$. Since any point may be taken as the point of projection this relation between the velocities holds at all points of the curve. *Liouville's Journal*, Tome IX. page 113.

274. The following example of Ossian Bonnet's theorem is important. It will be shown in the chapter on central forces that a particle P will describe an ellipse freely about a centre of force in one focus H_1, whose law of attraction is

μ_1/r_1^2, provided the velocity of projection at any point A is given by

$$v_1^2=\mu_1\left(\frac{2}{r_1}-\frac{1}{a}\right).$$

The same ellipse can also be described about a centre of force in the other focus H_2 whose law of attraction is μ_2/r_2^2 provided the velocity v_2 has the corresponding value. It immediately follows that *the particle can describe the ellipse freely about both centres of force acting simultaneously*, provided (1) the velocity v at any point A is given by

$$v^2=\mu_1\left(\frac{2}{r_1}-\frac{1}{a}\right)+\mu_2\left(\frac{2}{r_2}-\frac{1}{a}\right),$$

and (2) the direction of projection at A bisects externally the angle between the focal distances.

According to this mode of proof both the centres of force should be attractive, for it is evident that an ellipse could not be freely described about a single centre of repulsive force situated in either focus. But the law of continuity shows that this limitation is unnecessary. Supposing μ_1 and μ_2 to have arbitrary positive values, it has been proved that the equations of motion of a particle moving freely under both centres of force become satisfied when this value of v^2 is substituted in them. The equations contain only the first powers of μ_1 and μ_2 (see Art. 271) and can be satisfied only by the vanishing of the coefficients of these quantities. They will therefore still be satisfied if we change the signs of either μ_1 or μ_2.

In the same way we may introduce other changes into the theorem, provided always we can obtain a dynamical interpretation of the result.

275. *Ex.* 1. Prove that a particle can describe an ellipse freely under the action of three centres of force; one in each focus attracting as the inverse square and the third in the centre attracting as the direct distance. Find also the velocity of projection.

Ex. 2. Particles of masses m_1, m_2, &c. projected from the same point in the same direction with velocities u_1, u_2, &c. under the action of given forces F_1, F_2, &c. describe the same curve. Show that a particle of mass M projected in the same direction with a velocity V under the simultaneous action of all the forces F_1, F_2, &c. will also describe the same curve, provided

$$MV^2=m_1u_1^2+m_2u_2^2+\dots.$$

Ossian Bonnet, Note IV. to Lagrange's *Mécanique*.

Ex. 3. A bead is projected along a smooth elliptical wire under the action of two centres of force, one in each focus, and attracting inversely as the square of the distance. If TP, TQ be any two tangents to the ellipse, prove that the pressure when the bead is at P : pressure when the bead is at Q :: TQ^3 : TP^3.

Initial Tensions and radii of Curvature.

276. *Particles, of given masses, are connected together by inelastic rods or strings of given lengths and are projected in any given manner consistent with these constraints. It is required to find the initial values of the tensions and the radii of curvatures of the paths.*

The peculiarity of the problems on initial motion is that the velocities and directions of motion of all the particles are known. It will thus not be necessary to integrate the differential equations of motion, for the results of these integrations are given.

Supposing that there are n particles, we shall require besides the $2n$ equations of motion a geometrical equation corresponding to each reaction.

To show how the geometrical equations may be formed, let us suppose that two particles m_1, m_2 are connected by a rod or straight string of length l. The component velocities of the two particles in the direction of the string being necessarily equal, their *relative velocity* is the difference of their component velocities perpendicular to the rod; let these be V_1, V_2. If ϕ be the angle the rod makes with some fixed straight line, the geometrical equation is $l\dfrac{d\phi}{dt} = V_2 - V_1$.

The simplest method of obtaining the relative equations of motion is perhaps to reduce m_1 to rest. To effect this we apply to both particles (1) an acceleration equal and opposite to that of m_1, and (2) an initial velocity equal and opposite to that of m_1. The path of m_2 being now a circle whose centre is at m_1 and whose radius is l, the relative accelerations are those for a circular motion. (Art. 39.)

Let X_1, X_2 be the components along the rod of junction of all the forces and tensions which act on m_1, m_2 respectively. We then have (Art. 35)

$$-l\left(\frac{d\phi}{dt}\right)^2 = -\frac{(V_2 - V_1)^2}{l} = \frac{X_2}{m_2} - \frac{X_1}{m_1} \quad\text{...............(1).}$$

In this way we may form as many equations as there are reactions. By solving these the initial values of the reactions become known.

If the angular accelerations of the rods are also required, let Y_1, Y_2 be the component forces perpendicular to the rod which act on m_1, m_2. Then

$$l\frac{d^2\phi}{dt^2} = \frac{Y_2}{m_2} - \frac{Y_1}{m_1} \quad\text{......................(2).}$$

277. *To find the curvatures of the paths, we refer to the equations of motion in space.* The velocity and direction of motion of

each particle being known, we may conveniently use the tangential and normal resolutions. We thus have $2n$ equations of the form

$$m\frac{v^2}{\rho}=N,\quad m\frac{dv}{dt}=T\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where N, T are linear functions of the forces and tensions which act on the particle m.

These reactions having been found by considering the relative motion, we substitute in (3). The first of these determines the radius of curvature ρ of the path of m, and the second the tangential acceleration, if that be required.

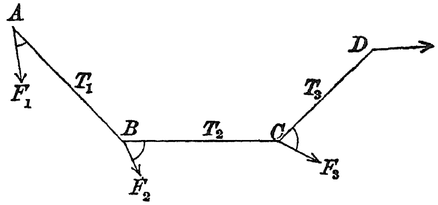
When any *one of the particles is constrained to describe a given curve*, the initial pressure of that curve is one of the unknown reactions. This pressure will be determined by the normal resolution of (3) since the radius of curvature of the path is the same as that of the constraining curve.

278. *If some or all the particles start from rest,* the equations of relative motion are simplified, for we then have $\phi'=0$ where the accent denotes d/dt. Since however *the direction of motion of a free particle at rest is not given, the tangential and normal resolutions are then inappropriate.* We can however use the Cartesian or polar resolutions in space. Since $\theta'=0$, the polar resolutions reduce to r'' and $r\theta''$ which are very simple forms. We must however bear in mind that if we require to differentiate the equations of motion this simplification must not be introduced until all the differentiations have been effected, Art. 281. We may also use Lagrange's equations, when the curvatures and not the tensions are required. These modifications of the general method are more especially useful in Rigid Dynamics and are discussed in the first volume of the author's treatise on that subject.

279. Examples. *Ex.* 1. Particles are attached to a string at unequal distances, and placed in the form of an unclosed polygon on a smooth table. The particles are then set in motion without impacts and are acted on by any forces. It is required to find the initial tensions and curvatures.

Let $ABCD$ &c. be any consecutive particles, and let the tensions of AB, BC, &c. be T_1, T_2, &c. Let the given forces be F_1, F_2, &c. and let them act in directions making angles α, β, &c. with AB, BC, &c. Let $l_1d\phi_1/dt$, $l_2d\phi_2/dt$, &c. stand for the known difference of the velocities of the consecutive particles resolved perpendicular to the rod or string joining them.

The particle B being reduced to rest, C is acted on by T_3/m_3 along CD, T_2/m_3 along CB, T_2/m_2 along CB, T_1/m_2 parallel to AB. Besides these there are the



impressed accelerating forces F_3/m_3 and $-F_2/m_2$. Since C describes a circle relatively to B, we have for the particle C

$$l_2\left(\frac{d\phi_2}{dt}\right)^2=\frac{T_3}{m_3}\cos C+T_2\left(\frac{1}{m_2}+\frac{1}{m_3}\right)+\frac{T_1}{m_2}\cos B+\frac{F_3}{m_3}\cos(C+\gamma)+\frac{F_2}{m_2}\cos\beta,$$

$$l_2\frac{d^2\phi}{dt^2}=\frac{T_3}{m_3}\sin C-\frac{T_1}{m_2}\sin B+\frac{F_3}{m_3}\sin(C+\gamma)+\frac{F_2}{m_2}\sin\beta,$$

where A, B, C, &c. are the internal angles of the polygon. The second resolution may be omitted if the angular accelerations of the several portions of string are not required.

An equation, corresponding to the first of these, can be written down for each of the n particles, beginning at either end, except the last. We thus form $(n-1)$ equations to find the $(n-1)$ tensions.

To find the initial radius of curvature of the path in space of any particle C we resolve along the normal to the path. Let the directions of motion of the particles be AA', BB', &c. and let v_1, v_2, &c. be the velocities of the particles. Then

$$\frac{m_3v_3^2}{\rho_3}=T_3\sin DCC'+T_2\sin BCC'-F_3\sin(DCC'-\gamma).$$

If the particle m_3 is initially at rest, $v_3=0$ and the last equation fails to determine ρ_3. The initial tensions may still be deduced from the first equation. The initial direction of motion of the particle coincides with the direction of the resultant force and is therefore known when the initial tensions have been found. The tangential acceleration is also known for the same reason. The determination of the radius of curvature requires further consideration.

Ex. 2. Heavy particles, whose masses beginning at the lowest are m_1, m_2, &c., are placed with their connecting strings on a smooth curve in a vertical plane. Find the initial tensions.

In this problem the arc between any two particles remains constant, so that the tangential accelerations of all the strings are equal. Let this common acceleration be f. Taking all the particles as one system, the tensions do not appear in the resulting equation, we have therefore

$$(m_1+m_2+\text{\&c.})f=-m_1g\sin\psi_1-m_2g\sin\psi_2-\text{\&c.},$$

where ψ_1, ψ_2, &c. are the angles the tangents at the particles make with the horizon.

Considering the lowest particle, we have

$$m_1f=-m_1g\sin\psi_1+T_1.$$

Considering the two lowest,

$$(m_1+m_2)f = -m_1 g \sin\psi_1 - m_2 g \sin\psi_2 + T_2,$$

and so on. Thus all the tensions T_1, T_2, &c. have been found.

If any tension is negative, that string immediately becomes slack. We also notice that the initial tensions are independent of the velocities of the particles.

To find the initial reactions, we use the normal resolutions. If v be the initial velocity of the particle m, we thus find $\dfrac{mv^2}{\rho} = -mg\cos\psi + R$.

Ex. 3. Three equal particles are connected by a string of length $a+b$ so that one of them is at distances a, b from the other two. This one is held fixed and the others are describing circles about it with the same angular velocity so that the string is straight. Prove that if the particle that was held fixed is set free the tensions in the two parts of the string are altered in the ratios $2a+b : 3a$ and $2b+a : 3b$. [Coll. Ex. 1897.]

Ex. 4. Three equal particles tied together by three equal threads are rotating about their centre of gravity. Prove that if one of the threads break, the curvatures of the paths instantaneously become 3/5, 6/5, 3/5ths respectively, of their former common value. [Coll. Ex. 1892.]

Ex. 5. Two particles are fastened at two adjacent points of a closed loop of string without weight which hangs in equilibrium over two smooth horizontal parallel rails. Prove that when the short piece of string between the particles is cut the product of the tensions before and after the cutting is equal to the product of the weights of the particles. [Coll. Ex. 1896.]

Ex. 6. Two particles of equal weight are connected by a string of length l which becomes straight just when it is vertical. Immediately before this instant the upper particle is moving horizontally with velocity $\sqrt{gl}$, and the lower is moving vertically downwards with the same velocity. Prove that the radius of curvature of the curve which the upper particle begins to describe is $\frac{5}{12}\sqrt{5}l$.

[Coll. Ex. 1897.]

Just after the impulse the upper particle begins to move in a direction inclined $\tan^{-1} 1/2$ to the horizon.

Ex. 7. Two equal particles A, B, are connected by a string of length l, the middle point C of which is held at rest on a smooth horizontal table. The particles describe the same circle on the table with the same velocity in the same direction, and the angle ACB is right. The point C being released, prove that the radii of curvature of their paths just after the string becomes tight are $5\sqrt{5}l/4$ and infinity.

Ex. 8. Four small smooth rings of equal mass are attached at equal intervals to a string, and rest on a smooth circular wire whose plane is vertical and whose radius is equal to one-third of the length of the string, so that the string joining the two uppermost is horizontal, and the line joining the other two is the horizontal diameter. If the string is cut between one of the extreme particles and the nearer of the middle ones, prove that the tension in the horizontal part of the string is immediately diminished in the ratio 9 : 5. [Coll. Ex. 1895.]

Ex. 9. Six equal rings are attached at equal intervals to points of a uniform weightless string, and the extreme rings are free to slide on a smooth horizontal rod. If the extreme rings are initially held so that the parts of the string

attached to them make angles α with the vertical, and then let go, the tension in the horizontal part of the string will be instantaneously diminished in the ratio of $\cos^2\alpha$ to $1+\sin^2\alpha$. [Coll. Ex. 1889.]

Ex. 10. Three particles A, B, C are in a straight line attached to points on a string and are moving in a plane with equal velocities at right angles to this line, their masses being m, m', m respectively. If B come in contact with a perfectly elastic fixed obstacle, prove that the initial radius of curvature of the paths which A and C begin to describe is $\frac{1}{4}a$, where $AB=BC=a$. [Coll. Ex. 1892.]

The particle B rebounds with velocity v. By considering the relative motion of A and B we have $4v^2/a=T/m$. By considering the space motion of A, $v^2/\rho=T/m$.

Ex. 11. A tight string without mass passes through two smooth rings A, B, on a horizontal table. Particles of masses p, q respectively are attached to the ends and a particle of mass m to a point O between A and B. If m be projected horizontally perpendicularly to the string, the initial radius of curvature ρ of its path is given by $(m+p+q)/\rho=p/a-q/b$, where $OA=a$, $OB=b$. [Coll. Ex. 1893.]

Ex. 12. A circular wire of mass M is held at rest in a vertical plane, on a smooth horizontal table, a smooth ring of mass m being supported on it by a string which passes round the wire to its highest point and from there horizontally to a fixed point to which it is attached. If the wire be set free, show that the pressure of the ring on it is immediately diminished by amount $\dfrac{2m^2g\sin^2\theta}{M+4m\sin^2\frac{1}{2}\theta}$, where θ is the angular distance of the ring from the highest point of the wire.

[Coll. Ex. 1897.]

Ex. 13. Two particles P, P' of masses m, m' respectively are attached to the ends of a string passing over a pulley A and are held respectively on two inclined planes each of angle α placed back to back with their highest edge vertically under the pulley. If each string makes an angle β with the plane, prove that the heavier particle will at once pull the other off the plane if

$$m'/m<2\tan\alpha\tan\beta-1. \qquad \text{[Coll. Ex. 1896.]}$$

Ex. 14. Two particles of masses m, M are attached at the points B, C of a string ABC, the end A being fixed. The two portions AB, BC rest on a smooth horizontal table, the angle at B being α. The particle M has a velocity communicated to it in a direction perpendicular to BC. Prove that if the strings remain tight, the initial radius of curvature of the locus of M is $a(1+n\sin^2\alpha)$, where $n=M/m$ and $BC=a$. [Coll. Ex. 1895.]

280. *To find the initial radius of curvature when the particle starts from rest.* In this problem it may be necessary to use differential coefficients of a higher order than the second. Let x, y be the Cartesian coordinates of a particle, then representing differential coefficients with regard to the time by accents

$$\rho=\frac{(x'^2+y'^2)^{\frac{3}{2}}}{x'y''-y'x''},$$

which takes a singular form when the component velocities x', y'

are zero. Putting $u = x'y'' - y'x''$, we have after differentiation

$$\begin{aligned} u' &= x'y''' - y'x''', \\ u'' &= x'y^{\text{iv}} - y'x^{\text{iv}} + x''y''' - y''x''', \\ u''' &= x'y^{\text{v}} - y'x^{\text{v}} + 2(x''y^{\text{iv}} - y''x^{\text{iv}}). \end{aligned}$$

For the sake of brevity let the initial value of any quantity be denoted by the suffix zero, thus x_0'' represents the initial value of x''. Using Taylor's theorem and remembering that $x_0' = 0$, $y_0' = 0$, we have

$$x'y'' - y'x'' = \tfrac{1}{2}(x_0''y_0''' - x_0'''y_0'')\, t^2 + \tfrac{1}{3}(x_0''y_0^{\text{iv}} - x_0^{\text{iv}}y_0'')\, t^3 + \&\text{c}.$$

Similarly $\qquad (x'^2 + y'^2)^{\frac{3}{2}} = (x_0''^2 + y_0''^2)^{\frac{3}{2}}\, t^3 + \&\text{c}.$

If the particle start from rest the initial radius of curvature is therefore zero. But if the circumstances of the problem are such that $x_0''y_0''' - x_0'''y_0'' = 0$, the radius of curvature is given by

$$\rho = \frac{3\,(x_0''^2 + y_0''^2)^{\frac{3}{2}}}{x_0''y_0^{\text{iv}} - x_0^{\text{iv}}y_0''}.$$

This is the general formula when the axes of x, y have any positions.

If the axis of y be taken in the direction of the resultant force $x_0'' = 0$, and if we then also have $x_0''' = 0$, the expression for the radius of curvature takes the simple form

$$\rho = 3\,\frac{y_0''^2}{x_0^{\text{iv}}}.$$

If Y_0 be the initial resultant force on the particle, X the transverse force, the formula when $X_0 = 0$, $X_0' = 0$ may be written

$$\rho = 3\,\frac{Y_0^2}{X_0''}.$$

The corresponding formula for ρ in polar coordinates may be obtained in the same way. We have when $r\,(r''\theta''' - r'''\theta'') = 0$ initially,

$$\frac{3\,(r^2\theta''^2 + r''^2)^{\frac{3}{2}}}{\rho} = 3r^2\theta''^3 + 6r''^2\theta'' + rr''\theta^{\text{iv}} - r\theta''r^{\text{iv}},$$

where the letters are supposed to have their initial values. If the initial value of $r'' = 0$, this takes the simpler form

$$3\left(\frac{1}{\rho} - \frac{1}{r}\right) = -\frac{r^{\text{iv}}}{r^2\theta''^2}.$$

281. Let n particles P_1, P_2, &c. at rest, be acted on by given forces and be connected by κ geometrical relations. To find the initial radius of curvature of the path of any one particle P we proceed in the following manner, though *in special cases a simpler process may be used.* We differentiate the dynamical equations twice and reduce each to its initial form by writing for all the coordinates (x_1, y_1), (x_2, y_2), &c. their initial values, and for (x_1', y_1'), &c. zero. We differentiate the geometrical equations four times and reduce each to its initial form. We then have sufficient equations to find the initial values of x'', x''', x^{iv}, &c., R, R', R'', &c. where R is any reaction. Lastly solving these for the coordinates of the particular particle under consideration we substitute in the standard formula for ρ.

This process may sometimes be shortened by eliminating the tensions (if these are not required) before differentiation. We thus avoid introducing their differential coefficients into the work.

282. Shorter Methods. We can sometimes simplify the geometrical relations by introducing subsidiary quantities, say θ, ϕ, &c. In this way we can express all the coordinates (x_1, y_1), &c. in terms of θ, ϕ, &c. by equations of the form

$$x=f(\theta, \phi, \&c.), \qquad y=F(\theta, \phi, \&c.) \qquad (1),$$

where θ, ϕ, &c. are independent variables. Substituting in the dynamical equations and eliminating the reactions, we have $2n-\kappa$ equations of the second order to determine θ, ϕ, &c. in terms of t. *These eliminations may be avoided and the results shortly written down by using Lagrange's equations.* Lagrange's method is described in chap. VII.

These equations, however obtained, contain θ, θ', θ''; ϕ, ϕ', ϕ'', &c. and by differentiation we can find as many higher differential equations as are required.

Since θ', ϕ', &c. are zero, we find by differentiation

$$x''=f_\theta\theta''+f_\phi\phi''+\dots,$$
$$x'''=f_\theta\theta'''+f_\phi\phi'''+\dots,$$
$$x^{iv}=3\left(f_{\theta\theta}\theta''^2+2f_{\theta\phi}\theta''\phi''+\dots\right)+f_\theta\theta^{iv}+f_\phi\phi^{iv}+\dots,$$

where suffixes as usual indicate partial differential coefficients, thus $f_\theta=df/d\theta$. There are similar expressions for the differential coefficients of y. Substituting in the standard form for ρ, we obtain the required radius of curvature.

283. We notice that if the partial differential coefficients f_θ, f_ϕ, &c. are zero the initial value of x^{iv} does not depend on any higher differential coefficients of θ, ϕ, &c., than the second, and these are given at once by the equations of motion. Since $\rho=3y''^2/x^{iv}$, when the axis of y is taken parallel to the resultant force on the particle, *the radius of curvature can then be found without differentiating the equations of motion.*

Since $$\frac{dx}{dt}=f_\theta\frac{d\theta}{dt}+f_\phi\frac{d\phi}{dt}+\dots,$$

the geometrical meaning of the equations $f_\theta=0$, $f_\phi=0$, &c. clearly is that $dx/dt=0$ for every geometrically possible displacement of the system. The point, whose initial radius of curvature is required, must begin to move parallel to the axis of y however the system is displaced.

284. Examples. *Ex.* 1. A particle is placed at rest at the origin and is acted on by forces X, Y parallel to the axes. If X, Y are expanded in powers of t and the lowest powers are $X=ft$, $Y=g$, show that the path near the origin is $y^3=mx^2$ and that the radius of curvature is zero. If $X=\frac{1}{2}ft^2$, $Y=g$, the path is a parabola whose radius of curvature is $3g^2/f$. We notice that in the first of these cases X' is finite, in the second zero.

Ex. 2. A particle is at rest on a plane, and forces X, Y in the plane begin to act on it. If these forces are functions of the coordinates x, y only, prove that the initial radius of curvature of the path is

$$3\left(X^2+Y^2\right)^{\frac{3}{2}}\Big/\left\{X\left(X\frac{dY}{dx}+Y\frac{dY}{dy}\right)-Y\left(X\frac{dX}{dx}+Y\frac{dX}{dy}\right)\right\}.$$

[Coll. Ex. 1895.]

This result follows from Art. 280.

Ex. 3. Two heavy particles are attached to two points B, C of a string, one end A being fixed. Prove that if the string ABC is initially horizontal, the initial radii of curvature of the paths of B and C are equal.

Prove also that if there are n particles on the horizontal string, all the initial radii of curvature are equal. If AB, BC were two equal heavy rods, hinged at B, and having A fixed, prove that the initial radii of curvature at B and C are unequal.

In this problem we see beforehand that it will be unnecessary to differentiate the equations of motion. Take the angles θ, ϕ, which the strings make with the initial position ABC as the independent variables, Art. 283.

Ex. 4. Two heavy particles P, Q, are connected by a string which passes through a smooth fixed ring O, the portions OP, OQ of the string making angles θ, ϕ, with the vertical. If the masses m, M of P, Q, satisfy the condition $m\cos\theta=M\cos\phi$, the initial radius of curvature of the path of P is given by

$$\frac{M+m}{M}\frac{\sin^2\theta}{\rho}=\frac{\sin^2\theta}{r}+\frac{\sin^2\phi}{l-r},$$

where $r=OP$ and l is the length of the string.

Take the polar equations of motion, eliminate the tension and differentiate twice. We thus find the initial values of θ'', r'', r^{iv}; since $r''=0$ the polar formula for ρ is much simplified.

Ex. 5. A uniform rod, moveable about one end O which is fixed, is held in a horizontal position by being passed through a small ring of equal weight; show that if the ring is initially at the middle point of the rod, when it is released the initial radius of curvature of its path is 9 times the length of the rod.

[Coll. Ex. 1887.]

Taking O as origin, the polar equation of motion of the particle shows that the initial values of r'', r''' are zero, while that of $r^{\text{iv}}=g\theta''+2r\theta''^2$. Taking moments

about O, Art. 261, we have $\frac{d}{dt}[(Mk^2+mr^2)\,\theta'] = (Ma+mr)\,g\cos\theta$. This gives the initial value of $\theta''=6g/7a$. The length of the radius of curvature follows by the differential calculus, Art. 280.

Ex. 6. Three particles whose masses are m_1, m_2, m_3 are placed at rest at the corners of a triangle ABC, and mutually attract each other with forces which vary according to some power of the distance. If $m_1m_2cF_3$, $m_2m_3aF_1$, $m_3m_1bF_2$ are the forces, prove that the initial radius of curvature ρ of the path of C is given by

$$\frac{3R^2}{\rho} = -m_2a\sin\phi\,\{-m_3F_1{}^2+m_1F_3\,(F_2-F_1)-PF_1{}'\}$$
$$+m_1b\sin\theta\,\{-m_3F_2{}^2+m_2F_3\,(F_1-F_2)-QF_2{}'\},$$

where θ, ϕ are the angles CA, CB make with the resultant force on C,

$$F_1{}'=dF_1/da, \quad F_2{}'=dF_2/db,$$
$$P=(m_2+m_3)\,aF_1+m_1\,(F_3c\cos B+F_2b\cos C),$$
$$Q=(m_1+m_3)\,bF_2+m_2\,(F_3c\cos A+F_1a\cos C),$$

and R is the resultant force on C.

Deduce that the initial radii of curvature of the three paths are infinite when the triangle is equilateral.

Small oscillations with one degree of freedom.

285. The theory of small oscillations has already been discussed in the chapter on Rectilinear Motion so far as systems with one degree of freedom are concerned. In this section a series of examples will be found showing the method of proceeding in cases somewhat more extended.

The particle, or system of particles, is supposed to be either in equilibrium or in some given state of motion. A slight disturbance being given, we express the displacements of the several particles at any subsequent time t from their positions in the state of equilibrium or motion by quantities x, y, &c These are supposed to be so small that their squares can be neglected. If required, corrections are afterwards introduced for the errors thus caused.

We form the equations of motion either by resolving and taking moments or by Lagrange's method. By neglecting the squares of the displacements these equations are made linear in x, y, z, &c. They are also linear in regard to the reactions between the several particles. Eliminating the latter we obtain linear equations which can in general be completely solved. The solution when obtained will enable us to determine whether the

system oscillates about its undisturbed state or departs widely from it on the slightest disturbance.

The principle of vis viva supplies an equation which has the advantage of being free from the unknown reactions, but it has the disadvantage that its terms contain the *squares* of the velocities, that is, the terms may be of the order we neglect. Being an accurate equation, it may sometimes be restored to the first order by differentiating it with regard to t and dividing by some small quantity. Generally the solution is more easily arrived at by using the equations of motion which contain the second differential coefficients with regard to t.

286. Examples. *Ex.* 1. Two particles whose masses are m, m' are connected by a string which passes through a small hole in a smooth horizontal table. The particle m' hangs vertically, while m is projected on the table perpendicularly to the string with such a velocity that m' is stationary. If a small disturbance is given to the system so that m' makes vertical oscillations, prove that the period is $2\pi\sqrt{\dfrac{(m+m')\,c}{mg}}$ where c is the mean radius vector of the path of m.

Let r, θ be the polar coordinates of m, z the depth of m', l the length of the string and T the tension. The equations of motion after the disturbance are

$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=-\frac{T}{m},\qquad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=0,$$

$$\frac{d^2z}{dt^2}=g-\frac{T}{m'},\qquad\qquad r+z=l.$$

The second equation gives $r^2d\theta/dt=h$, where h is a constant whose magnitude depends on the disturbance. Eliminating T, z and $d\theta/dt$ we find

$$(m+m')\frac{d^2r}{dt^2}-\frac{mh^2}{r^3}=-m'g.$$

Let $r=c+\xi$ where c is a constant which is as yet arbitrary except that the variable ξ is so small that its square can be neglected.

$$\therefore\ (m+m')\frac{d^2\xi}{dt^2}+m\frac{3h^2}{c^4}\xi=\frac{mh^2}{c^3}-m'g.$$

Let us now choose c to be such that the right-hand side of the equation is zero; then $mh^2=m'c^3g$. Substituting for h we find

$$\xi=A\sin(nt+\alpha),\quad n^2=\frac{3m'}{m+m'}\,\frac{g}{c}.$$

Since ξ is wholly periodic and has no constant term, its mean value is zero, when taken either for any long time or for the period of oscillation. It follows that $r=c$ is the mean radius vector of the path of m after the disturbance. This is not necessarily the same as the radius of the circle described before disturbance; whether it is so or not depends on the nature of the disturbance given to the system.

Let the particle m before disturbance be describing a circle of radius a with velocity V, then $mV^2/a=m'g$, each being the tension of the string; and the angular

momentum of m is mVa. If the disturbance be given by a vertical blow B applied to the particle m', this reacts on m by an impulsive tension, and, the moment of this about O being zero, the angular momentum of m is unaltered. In this case we have $h=Va$ and we find $c=a$. If the disturbance be given by a transverse blow B applied at m, the velocity of m is changed to V' where $V'-V=B/m$. In this case $h=V'a$ and c is not equal to a.

Ex. 2. A particle of mass m is attached to two points A, B by two elastic strings each having the same modulus E and natural length l. If the particle be displaced parallel to this line, prove that the time of oscillation is $2\pi\sqrt{(ml/2E)}$. [Coll. Ex. 1895.]

Ex. 3. A heavy particle hangs in equilibrium suspended by an elastic string whose modulus is three times the weight of the particle. The particle is slightly displaced in a direction making an angle $\cot^{-1}4$ with the horizontal and is then released. Prove that the particle will oscillate in an arc of a small parabola terminated by the ends of the latus rectum. [Math. Tripos, 1897.]

Ex. 4. A straight rod AB without weight is in a vertical position, with its lower end A hinged to a fixed point, and a weight attached to the upper end B. To B are attached three similar elastic strings equally stretched to a length k times their natural length and equally inclined to one another, their other ends being attached to three fixed points in the horizontal plane through B. Show that, when the strings obey Hooke's law, the condition for stability of equilibrium is that the weight must not exceed that which, when suspended by one of the strings, would cause an increase of length equal to $\frac{3}{2}(2-1/k)AB$. Show that, when this condition is fulfilled, the system can perform small vibrations parallel to any vertical plane. [Math. Tripos, 1888.]

Ex. 5. A smooth ring P can slide freely on a string which is suspended from two fixed points A and B not in the same horizontal line. If P be disturbed, find the time of a small oscillation in the vertical plane passing through A and B. If T be the time, $(T/2\pi)^2g=4(rr')^{\frac{3}{2}}/(r+r')\{(r+r')^2-4c^2\}^{\frac{1}{2}}$, where r, r' are the distances AP, BP in equilibrium and $AB=2c$.

Ex. 6. A rod of mass M hangs in a horizontal position supported by two equal vertical elastic strings, modulus λ and natural length a. Prove that if the rod receive a small displacement parallel to itself, the period of a horizontal oscillation is $2\pi\left(\frac{a}{g}+\frac{Ma}{2\lambda}\right)^{\frac{1}{2}}$. [Coll. Ex. 1897.]

Ex. 7. A particle of mass m is attached to an elastic string stretched between two points fixed in a smooth board of mass M, and the board is free to slide on a smooth table. Prove that the period in which the particle oscillates is less than it would be if the board were fixed in the ratio $1:\sqrt{(1+m/M)}$. [Coll. Ex. 1895.]

Reduce the board to rest.

Ex. 8. A ring of mass nm is free to slide on a smooth horizontal wire, and a string tied to it passes through a small ring vertically below the wire at a depth h, and supports a particle of mass m. Prove that if the first mass be released when the upper part of the string makes an angle α with the vertical, and if θ be the inclination after a time t, the equation of motion is

$$h(n+\sin^2\theta)(d\theta/dt)^2=2g\cos^4\theta(\sec\alpha-\sec\theta).$$

Prove hence that the small oscillations about the position of equilibrium will be synchronous with a simple pendulum of length nh. [Coll. Ex. 1896.]

Ex. 9. A crane is lowering a heavy body and the chain is paid out with a uniform velocity V. Prove that the small lateral oscillations of the body are determined by

$$r\frac{d^2\theta}{dr^2}+2\frac{d\theta}{dr}+\frac{g\theta}{V^2}=0,$$

where r is the length of the chain at any time and θ its inclination to the vertical, the weight of the chain being neglected.

Also if $\theta\sqrt{r}=y$, $2\sqrt{gr}=xV$, prove that

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-1)\,y=0.$$

This equation can be solved by the use of Bessel's functions. See Gray and Mathews' *Treatise on Bessel's Functions*. [Coll. Ex. 1895.]

Ex. 10. A gravitating solid of revolution is cut by a plane perpendicular to the axis. A particle is fastened by a fine string of length l to a point in the prolongation of the axis, so that when the string is perpendicular to the plane section the particle just does not touch the plane at its centre O. Assuming the conditions such that when the particle is slightly disturbed the motion is that of a simple pendulum, prove that the time T of a small oscillation is given by $l(2\pi/T)^2=R+\frac{1}{2}lR'$ where R is the force exerted by the solid on a unit mass at O and R' is the space variation of the force at O, taken outside the solid, along the axis. [Coll. Ex. 1892.]

Small oscillations with two or more degrees of freedom.

287. Oscillations about equilibrium. *A particle is in equilibrium under the action of forces X, Y which are given functions of the coordinates. A slight disturbance being given, it is required to determine whether the particle oscillates and the nature of the motion.*

Let a, b be the coordinates of the position of equilibrium, $a+x$, $b+y$, the coordinates at any time t. We shall assume as the standard case that x and y are small throughout the motion. Solving the equations of motion we shall express x, y in terms of t. By examining the results we shall determine whether and how nearly the subsequent motion follows the standard form.

We shall suppose that the forces X, Y can be expanded in integer powers of x, y, viz.

$$X=Ax+By, \quad Y=B'x+Cy \quad\quad (1),$$

where we have rejected the higher powers in our first approximation. There are no constant terms because X, Y vanish in the position of equilibrium. Taking the mass of the particle as unity, the equations of motion are

$$\frac{d^2x}{dt^2}=Ax+By, \quad \frac{d^2y}{dt^2}=B'x+Cy \quad\quad (2).$$

To solve these we let δ represent d/dt,

$$\therefore\ (\delta^2 - A)x - By = 0, \quad -B'x + (\delta^2 - C)y = 0 \ldots(3).$$

Eliminating y, we have the two forms

$$\begin{vmatrix} \delta^2 - A, & -B \\ -B', & \delta^2 - C \end{vmatrix} x = 0, \quad By = (\delta^2 - A)x \ldots\ldots\ldots(4).$$

The first of these is a differential equation with constant coefficients. Its solution can be written down by the usual rules given in treatises on differential equations. The solution contains four arbitrary constants, and the value of y follows from that of x, without the introduction of any new constants.

The usual method is to assume as a trial solution $x = Le^{mt}$. Substituting we arrive at the biquadratic

$$m^4 - (A + C)m^2 + AC - BB' = 0 \ldots\ldots\ldots\ldots(5);$$

$$\therefore\ m^2 = \tfrac{1}{2}[A + C \pm \sqrt{\{(A - C)^2 + 4BB'\}}].$$

Assuming that no two roots are equal, let the four values of n be $\pm m$, $\pm n$; then

$$x = L_1 e^{mt} + L_2 e^{-mt} + M_1 e^{nt} + M_2 e^{-nt} \ldots\ldots\ldots\ldots(6),$$

where L_1, L_2 &c. are four arbitrary constants and the values of m may be real or imaginary.

It is at once obvious, if m be positive or of the form $r \pm p\sqrt{-1}$, where r is positive, that the value of x will become large by efflux of time. *It is therefore necessary for an oscillatory motion that all the real roots and the real parts of the imaginary roots of the determinantal equation* (5) *should be negative.*

Since the sum of the four roots of (5) is zero, some of the real parts must be positive unless the four roots are of the form $\pm p\sqrt{-1}$. *It is therefore necessary for an oscillatory motion that both the roots of the quadratic* (5) *should be real and negative.* The algebraical conditions for this are, that both $(A - C)^2 + 4BB'$ and $AC - BB'$ should be positive and $A + C$ negative.

As our solution represents the motion only when x and y remain small, it is unnecessary for us here to consider any case except that in which the roots of (5) take the forms $m^2 = -p^2$, $n^2 = -q^2$. The motion is then given by

$$\left.\begin{aligned} x &= L \sin(pt + \alpha) + M \sin(qt + \beta) \\ y &= L' \sin(pt + \alpha) + M' \sin(qt + \beta) \end{aligned}\right\} \ldots\ldots\ldots\ldots(7),$$

where $BL' = -(p^2 + A)L$ and $BM' = -(q^2 + A)M$. The quantities p^2, q^2 are the roots of

$$(p^2 + A)(p^2 + C) - BB' = 0 \text{..................} (8).$$

288. *If B, B' have the same sign, the roots of the quadratic* (8) *are separated by each of the values $p^2 = -A$, $p^2 = -C$.* To prove this, it is sufficient to notice that the left-hand side of that equation is positive when $p^2 = \pm\infty$ and is negative when p^2 has either of the separating values.

It is also sometimes useful to notice that the roots cannot be equal unless the two separating values A and C are equal and that the equal roots are then $p^2 = -A = -C$. If $AC - BB' = 0$ the biquadratic (5) has two equal zero roots, though the roots of the same equation regarded as a quadratic are unequal.

289. *To find the four arbitrary constants L, M, α, β, we solve the equations* (7) *with regard to the trigonometrical terms.* We thus find

$$\left.\begin{aligned} By + (q^2 + A)x &= -(p^2 - q^2)\, L \sin(pt + \alpha) \\ By + (p^2 + A)x &= (p^2 - q^2)\, M \sin(qt + \beta) \end{aligned}\right\} \text{..................} (9).$$

Putting $t = 0$, we at once have the values of $L \sin\alpha$, $M \sin\beta$ in terms of the initial values of the coordinates. Differentiating with regard to t and again putting $t = 0$, we find $L \cos\alpha$, $M \cos\beta$ in terms of the initial velocities.

290. Equal roots. The case in which the equation (5) has equal roots has been excepted. This occurs when either $(A - C)^2 + 4BB' = 0$ or $AC - BB' = 0$. When B, B' have the same sign the first alternative requires $A = C$ and either B or B' equal to zero. In the second alternative the equation has two zero roots.

Excepting when both B and B' are zero, the solution of the dynamical equations (2) is known to contain terms of the form $(Lt + L')\, e^{mt}$. If m is positive or zero (or has its real part positive or zero), this term will increase indefinitely with t. If however the real part of m is negative and not zero, say equal to $-r$, the maximum value of Lte^{-rt} is L/re. Since L is so small that its square can be neglected, this term in the solution will always remain small except when r also is small. *The existence of equal roots in the determinantal equation* (5) *does not therefore necessarily imply that the oscillation becomes large.*

291. Before disturbance the particle P was in equilibrium at the origin under the influence of the forces X, Y given by (1) Art. 287. When $AC = BB'$, the equations $X = 0$, $Y = 0$ are satisfied by values of x, y other than zero. These lie on the straight line $Ax + By = 0$. *The dynamical significance of the condition $AC = BB'$ is therefore that there are other positions of equilibrium in the immediate neighbourhood of the origin.* The roots of equation (8) being $p^2 = 0$, $q^2 = -A - C$, the values of x, y take the form

$$x = L_1 t + L_2 + M \sin(qt + \beta),$$
$$By = -A\,(L_1 t + L_2) - CM \sin(qt + \beta).$$

The first terms represent a uniform motion along the line of equilibrium, while the trigonometrical terms represent an oscillation in the direction $By = -Cx$. Whether the particle will travel far or not along the line of equilibrium will depend on the nature of the forces when x, y become large.

292. Principal oscillations. Let the type of motion be that represented by such equations as (7). By giving the particle the proper initial conditions it may be made to move in either of the ways defined by the following partial solutions

$$x = L \sin(pt + \alpha), \quad y = L' \sin(pt + \alpha) \quad \text{.........(10)},$$
$$x = M \sin(qt + \beta), \quad y = M' \sin(qt + \beta) \quad \text{.........(11)}.$$

Each of these is called a principal oscillation and all the modes of oscillation included in (7) are compounded of these two. *The dynamical peculiarity of a principal oscillation is the singleness of the period.*

The solution (10) is sometimes taken as the trial solution instead of the exponential used in obtaining (5). Practically we then begin the solution by finding the principal oscillations and finally combine these into the general solution (7).

The paths of the particle when describing the principal oscillations are the two straight lines

$$Ly = L'x, \quad My = M'x \quad \text{..................(12)}.$$

In each oscillation the ratio of the coordinates, being equal to L'/L or M'/M, is constant throughout the motion. We have by (7), using the values of $p^2 + q^2$, p^2q^2, given by the coefficients of the quadratic (8),

$$\frac{L'M'}{LM} = \frac{(p^2 + A)(q^2 + A)}{B^2} = -\frac{B'}{B} \quad \text{..............(13)}.$$

It follows that when B, B' have the same sign, the ratios L'/L, M'/M have opposite signs. In one principal oscillation, the coordinates x, y increase together; in the other, when one increases the other decreases.

We also notice that when $B' = B$, the two straight lines (12) are at right angles.

The directions of these rectilinear oscillations may be obtained without investigating the motion. The lines must be so placed that if the particle be displaced along either, the perpendicular force must be zero. The lines are therefore given by

$$Xy - Yx = 0; \quad \therefore By^2 + (A - C)xy - B'x^2 = 0.$$

These lines are real when $(A - C)^2 + 4BB'$ is positive. This condition is satisfied when the roots of the determinantal equation (5) are real or of the form $p\sqrt{-1}$.

293. When the coordinates are such that only one varies along each principal oscillation, they are called *principal coordinates.*

Referring to the equations (9), we see that if we put

$$By + (q^2 + A)x = \xi, \quad By + (p^2 + A)x = \eta,$$

ξ, η will be the principal coordinates. This transformation of coordinates is always possible, so long as p^2 and q^2 are real and unequal.

We may also discover the principal coordinates without previously finding the values of p^2, q^2. We deduce from the equations (2)

$$\frac{d^2}{dt^2}(x + \lambda y) = (A + \lambda B')\left(x + \frac{B + \lambda C}{A + \lambda B'}y\right).$$

by using an indeterminate multiplier λ. If now we write $(B+\lambda C)/(A+\lambda B') = \lambda$, we see that $x + \lambda y$ will be a trigonometrical function with one period. We have a quadratic to find λ; representing the roots by λ_1, λ_2, the principal coordinates are $\xi = x + \lambda_1 y$, $\eta = x + \lambda_2 y$, or any multiples of these.

294. Conservative forces. *When the forces which act on the particle are conservative,* the solution admits of some simplifications. Let U be the force function, then, since dU/dx and dU/dy vanish in the position of equilibrium, we have by Taylor's theorem,

$$U = U_0 + \tfrac{1}{2}(Ax^2 + 2Bxy + Cy^2) + \dots \quad \dots\dots\dots\dots(1).$$

It follows that the equations of motion are

$$\frac{d^2x}{dt^2} = X = Ax + By, \quad \frac{d^2y}{dt^2} = Y = Bx + Cy \dots\dots\dots(2).$$

Comparing these with the former values of X, Y, we see that $B' = B$.

If we turn the axes round the origin we know by conics that the equation (1) can be always cleared of the term containing the product xy. Representing the new coordinates by ξ, η, let the expression for U become

$$U = U_0 + \tfrac{1}{2}(A'\xi^2 + C'\eta^2) + \dots \quad \dots\dots\dots\dots\dots(3),$$

where $A' + C' = A + C$, $A'C' = AC - B^2$. The equations of motion are then

$$\frac{d^2\xi}{dt^2} = A'\xi, \quad \frac{d^2\eta}{dt^2} = C'\eta \quad \dots\dots\dots\dots\dots\dots\dots(4).$$

The motion is oscillatory for all displacements or for none according as A', C' are both negative or both positive. If A' is negative and C' positive, the motion is oscillatory for a displacement along the axis of ξ and not wholly oscillatory for other displacements.

The level curves of the field of force are obtained by equating U to a constant; in the neighbourhood of the position of equilibrium, these become the conics

$$Ax^2 + 2Bxy + Cy^2 = N, \text{ or } A'\xi^2 + C'\eta^2 = N.$$

The lines of the principal oscillations are the directions of the principal diameters of the limiting level conic, and the periods of the principal oscillations are proportional to the lengths of the diameters along which the particle moves.

295. The representative particle. The investigation of the small oscillations of a particle in a given field of force has a more extended application to dynamical problems than appears at first sight. Suppose, for example, that a system, consisting of several particles connected together by geometrical relations, has two degrees of freedom. Let the position of this system be defined by the two coordinates x, y. The equations giving the small oscillations, after the elimination of the reactions, take the form

$$\frac{d^2x}{dt^2} = Ax + By, \qquad \frac{d^2y}{dt^2} = B'x + Cy,$$

because the squares of x and y are neglected. If $B = B'$ these are the equations of motion of a single particle moving in the field of force defined by

$$U - U_0 = \tfrac{1}{2}(Ax^2 + 2Bxy + Cy^2).$$

The investigations given in Art. 292 and Art. 294 apply therefore to both problems.

To exhibit the motion of an oscillating system to the eye, we take its coordinates x, y to be also the Cartesian coordinates of an imaginary particle which moves freely in the field of force U. We represent by a figure the level conics, the path of this representative particle, and sketch the positions of the principal oscillations. The special peculiarities of the motion will then become apparent in the figure.

296. Test of stability*. Let the field of force in which the particle moves be given by the function U. Since dU/dx and dU/dy vanish in the position of equilibrium, U must be at that point a maximum or a minimum. In the neighbourhood we have

$$U - U_0 = \tfrac{1}{2}(Ax^2 + 2Bxy + Cy^2) + \ldots$$

If $AC - B^2$ is positive, U is a maximum or a minimum for all displacements according as the common sign of A and C is negative or positive, and if $AC - B^2$ is negative, U is a maximum for

* The energy test of the stability of a position of equilibrium is given by Lagrange in the *Mécanique Analytique.* He gives both this proof and that in Art. 297. The demonstration for the general case of a system of bodies has been much simplified by Lejeune-Dirichlet in *Crelle's Journal,* 1846, and *Liouville's Journal,* 1847. See the author's *Rigid Dynamics,* vol. I.; the corresponding test for the stability of a state of motion is in vol. II.

some and a minimum for other displacements. It follows from Art. 294 that *the motion of the particle, when disturbed from its position of equilibrium, will be wholly oscillatory if U is a real maximum at that point. The particle will oscillate for some displacements and not for others if U has a stationary value, and will not oscillate for any displacement if U is a real minimum.*

We have here assumed that *all* the coefficients A, B, C are not zero. When this happens the cubic terms in the expression for U govern the series. The equations of motion (2) of Art. 295 will then have terms of the second order of small quantities on their right-hand sides.

Besides this if $AC - B^2 = 0$, the quadratic terms of the expression for U take the form of a perfect square, viz. $(Ax + By)^2/A$. In this case the forces $X = dU/dx$ and $Y = dU/dy$ contain the common factor $Ax + By$ so that there are other positions of equilibrium in the neighbourhood of the origin, see Art. 291. To determine the motion, even approximately, it is necessary to take account of the powers of x, y of the higher orders.

The geometrical theory of maxima and minima has a corresponding peculiarity, for it is shown in the Differential Calculus that further conditions, involving the higher powers, are necessary for a maximum or minimum.

The following investigation shows how far this correspondence extends.

297. Let a particle be in equilibrium at a point P_0 whose coordinates are x_0, y_0, and let $U = f(x, y)$ be the work function. Let the particle be projected with a small velocity v_1 from a point P_1, whose coordinates are x_1, y_1, very near to P_0. The equation of vis viva gives (Art. 246)

$$v^2 = v_1^2 + 2(U - U_1) \quad \text{.......................(1)},$$

$$= v_0^2 + 2(U - U_0) \quad \text{.......................(2)},$$

where

$$v_0^2 = v_1^2 + 2(U_0 - U_1) \quad \text{.......................(3)}.$$

Let U be a maximum at the point P_0 for all directions of displacement, then $U_1 < U_0$ and v_0^2 is a small positive quantity. As the particle recedes from P_0, $U_0 - U$ increases, but the equation (2) shows that the particle cannot go so far that $U_0 - U$ becomes

greater than the small quantity $\frac{1}{2}v_0^2$. The equilibrium is therefore stable for displacements in all directions.

Let U be a minimum at P_0 for all directions of displacement, then as the particle moves from P_0 the difference $U - U_0$ increases. So far as the principle of vis viva is concerned, there is nothing to prevent the particle from receding indefinitely from P_0.

Let U be a maximum for some directions of displacement and a minimum for others. The particle cannot recede far from P_0 in the directions for which U is a maximum, but there is nothing to restrict the motion in the other directions.

298. *Ex.* A particle P is in equilibrium under the action of a system of fixed attracting bodies situated in one plane, the law of attraction being the inverse κth power of the distance. Prove that, if $\kappa > 1$, the equilibrium of P cannot be stable for all displacements in that plane, though it may be stable for some and unstable for other displacements. If $\kappa < 1$, the equilibrium cannot be unstable for all displacements in that plane.

To prove this let m_1 be any particle of the attracting mass, coordinates f, g; let x, y be the coordinates of P. The potential of m_1 at P is by definition $U_1 = \dfrac{m_1}{(\kappa - 1) r_1^{\kappa - 1}}$, where r_1 is the distance of m_1 from P. We then find by a partial differentiation

$$\frac{d^2U_1}{dx^2} + \frac{d^2U_1}{dy^2} = \frac{(\kappa - 1) m_1}{r_1^{\kappa+1}}.$$

Summing this for all the particles of the attracting mass and writing $U = \Sigma U_1$, we find

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} = (\kappa - 1) \Sigma \frac{m}{r^{\kappa+1}}.$$

The right-hand side is positive or negative according as $\kappa > 1$ or $\kappa < 1$.

Taking the equilibrium position of P for the origin and the principal directions of motions for the axes, Art. 294, we see by Taylor's Theorem

$$U = U_0 + \tfrac{1}{2}(A'\xi^2 + C'\eta^2) + \ldots,$$

where $A' = d^2U/dx^2$, $C' = d^2U/dy^2$. It is evident that U cannot be a maximum for all displacements in the plane of xy if $A' + C'$ is positive and cannot be a minimum for all displacements in the plane if this sum is negative. The result also follows from Art. 296.

299. Barrier curves. It is clear that this line of argument may be extended to apply to cases in which there is no given position of equilibrium in the neighbourhood of the point of projection. Let the particle be projected from any point P_1 with any velocity v_1 in any direction. Throughout the subsequent motion we have

$$v^2 = v_1^2 + 2(U - U_1),$$

where U is a given function of x, y and U_1 is its value at the point of projection.

If we equate the right-hand side of this equation to zero, we obtain the equation of a curve traced on the field of force at which the velocity of the particle, if it arrive there, is zero. *This curve is therefore a barrier to the motion, which the particle cannot pass.*

If the barrier curve be closed as in Art. 297, the particle is, as it were, imprisoned, and cannot recede from its initial position beyond the limits of the curve. Some applications of this theorem will be given in the chapter on central forces.

The right-hand side of the equation will in general have opposite signs on the two sides of the barrier. When this is the case the particle, if it reach the barrier in any finite time, must necessarily return, because the left-hand side of the equation cannot be negative.

If the right-hand side of the equation have the same sign on both sides of the barrier, that sign must be positive, and U must be a minimum at all points of the barrier. The particle is therefore approaching a position of equilibrium and arrives there with velocity equal to zero. The particle therefore will remain on the barrier, see Art. 99.

The barrier is evidently a level curve of the field of force and, as the particle approaches it, the resultant force must be normal to the barrier. Just before the particle arrives at its position of zero velocity, the tangential component of the velocity must be zero, for this component cannot be destroyed by the force. The path cannot therefore touch the barrier, but must meet it perpendicularly or at a cusp.

300. Examples. *Ex.* 1. Two heavy particles of masses m, m', are attached to the points A, B of a light elastic string. The upper extremity O is fixed and the string is in equilibrium in a vertical position. A small vertical disturbance being given, find the oscillations.

Let x, y be the depths of m, m' below O; a, b the unstretched lengths of OA, AB, E the coefficient of elasticity. The equations of motion reduce to

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} + \left(\frac{E}{a} + \frac{E}{b}\right)x - \frac{E}{b}y = mg \\ -\frac{E}{b}x + m'\frac{d^2y}{dt^2} + \frac{E}{b}y = m'g + E \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

To solve these we put

$$x-h=L\sin(pt+a), \qquad y-k=M\sin(pt+a) \quad\text{..........(2)},$$

the constants h, k being introduced to cancel the right-hand sides of the equations of motion. Since $x=h$, $y=k$ make $d^2x/dt^2=0$, $d^2y/dt^2=0$, these constants are the equilibrium values of x, y. We then find

$$\left(mp^2-\frac{E}{a}-\frac{E}{b}\right)\left(m'p^2-\frac{E}{b}\right)=\frac{E^2}{b^2}, \qquad \frac{L}{M}=1-\frac{m'b}{E}p^2 \quad\text{..............(3)}.$$

One principal oscillation is given by (2) and the other by using instead of p^2, the other root of the quadratic. It follows that in one oscillation the two particles are always moving in the same directions, that is both are moving upwards or both downwards. In the other when one moves upwards the other moves downwards.

Ex. 2. Two heavy particles, of masses m, M, are attached to the points A, B of a light inextensible string, the upper extremity O being fixed. Prove that the periods of the small lateral oscillations are $2\pi/p$ and $2\pi/q$ where p and q are the roots of

$$\frac{1}{p^4}-\frac{a+b}{g}\,\frac{1}{p^2}+\frac{m}{M+m}\,\frac{ab}{g^2}=0,$$

and $OA=a$, $AB=b$. Prove also that the magnitudes of the principal oscillations in the inclinations of the upper and lower strings to the vertical are in the ratio $(g-bp^2)/ap^2$. Show that in one principal oscillation the two particles are on the same side of the vertical through O and in the other on opposite sides.

Ex. 3. Two particles M, m, are connected by a fine string, a second string connects the particle m to a fixed point, and the strings hang vertically; (1) m is held slightly pulled aside a distance h from the position of equilibrium, and, being let go, the system performs small oscillations; (2) M is held slightly pulled aside a distance k, without disturbance of m, and being let go the system performs small oscillations. Prove that the angular motion of the lower string in the first case will be the same as that of the upper string in the second if $Mk=(M+m)h$.

[Math. Tripos, 1888.]

Ex. 4. Three beads, the masses of which are m, m', m'', can slide along the sides of a smooth triangle ABC and attract each other with forces which vary as the distance. Find the positions of equilibrium and prove that if slightly disturbed the periods $2\pi/p$ of oscillation are given by

$$\begin{aligned}(p^2-\alpha)(p^2-\beta)(p^2-\gamma)-m'm''(p^2-\alpha)\cos^2 A-m''m(p^2-\beta)\cos^2 B&\\ -mm'(p^2-\gamma)\cos^2 C-2mm'm''\cos A\cos B\cos C&=0,\end{aligned}$$

where α, β, γ represent $m''+m'$, $m+m''$, $m'+m$ respectively.

Ex. 5. A particle P of unit mass is placed at the centre of a smooth circular horizontal table of radius a. Three strings, attached to the particle, pass over smooth pulleys A, B, C at the edge of the table and support three particles of masses m_1, m_2, m_3; the pulleys being so placed that the particle P is in equilibrium. A small disturbance being given, prove that the periods of the oscillations are $2\pi/p$, where

$$\left\{\frac{p^2(1+\sigma)}{p^2+g/a}\right\}^2-\frac{p^2(1+\sigma)\,\sigma}{p^2+g/a}+\frac{\sigma^2 H}{4m_1m_2m_3}=0,$$

$$H=(m_1+m_2-m_3)(m_2+m_3-m_1)(m_1+m_3-m_2),$$

$$\sigma=m_1+m_2+m_3.$$

Ex. 6. A heavy particle P is suspended by a string of length l to a point A which describes a horizontal circle of radius a with a slow angular velocity n. Prove that the two periods of the oscillatory motion are $2\pi/n$ and $2\pi\sqrt{l/g}$.

301. Particle on a surface. *Ex.* 1. A heavy particle rests in equilibrium on the inside of a fixed smooth surface at a point O, at which the surface has only one tangent plane. The particle being slightly disturbed, it is required to find the oscillations.

Taking the point O as origin and the tangent plane as the plane of xy, the equation of the surface may be written

$$z=\tfrac{1}{2}(ax^2+by^2)+\dots,$$

where the axes of x, y are the tangents to the principal sections and $1/a$, $1/b$ are the radii of curvature of those sections. By the principles of solid geometry the direction cosines of the normal at any point P become $(ax, by, 1)$ when the squares of x, y are neglected. The equations of motion are therefore

$$m\frac{d^2x}{dt^2}=-Rax, \qquad m\frac{d^2y}{dt^2}=-Rby, \qquad m\frac{d^2z}{dt^2}=-mg+R.$$

Since z is of the second order of small quantities the third equation shows that $R=mg$, and the other two become

$$\frac{d^2x}{dt^2}=-agx, \qquad \frac{d^2y}{dt^2}=-bgy.$$

If a and b are positive, that is if both the principal sections are concave upwards, the motion is oscillatory and the two periods of oscillations are $2\pi/\sqrt{ag}$ and $2\pi/\sqrt{bg}$. The particle, by definition, performs a principal oscillation when its motion has but one period. This occurs when

$$(1)\quad x=0,\ y=\quad \sin(\sqrt{bg}t+\beta), \qquad (2)\quad y=0,\ x=A\sin(\sqrt{ag}t+\alpha).$$

The directions of these oscillations are the tangents to the principal sections.

Ex. 2. A particle rests on a smooth surface which is made to revolve with uniform angular velocity ω about the vertical normal which passes through the particle. Show that the equilibrium is stable (1) if the curvature is synclastic upwards, and ω does not lie between certain limits, or (2) if the curvature is anticlastic and the downward principal radius is greater than the upward principal radius, and ω exceeds a certain limit. Find the limits of ω in each case.

[Math. Tripos, 1888.]

Taking as axes the tangents to the principal sections, the equations of motion (Art. 227) reduce to

$$\frac{d^2x}{dt^2}-\omega^2x-2\omega\frac{dy}{dt}=-gax, \qquad \frac{d^2y}{dt^2}-\omega^2y+2\omega\frac{dx}{dt}=-gby.$$

To solve these we put $x=L\sin(pt+\alpha)$, $y=L'\cos(pt+\alpha)$. We then obtain a quadratic for p^2 and the ratio L'/L.

The path of the particle relatively to the moving surface when performing the principal oscillation defined by either value of p^2 is the ellipse $\left(\frac{x}{L}\right)^2+\left(\frac{y}{L'}\right)^2=1$. The two ellipses are coaxial.

302. The insufficiency of the first approximation. In forming the equations of motion in Arts. 287, 294, we have rejected the squares of x and y.

But unless the extent of the oscillation is indefinitely small, the rejected terms have some values, and it may be, that they sensibly affect the results of the first approximation. See Art. 141.

303. To find a second approximation we include in the equations (2) of Art. 287 the terms of the second order. We write these in the form

$$\left.\begin{aligned}(\delta^2-A)\,x-By&=E_1x^2+2E_2xy+E_3y^2\\-B'x+(\delta^2-C)\,y&=F_1x^2+2F_2xy+F_3y^2\end{aligned}\right\}\quad\text{.....................}\quad(1).$$

Taking as our first approximation

$$\left.\begin{aligned}x&=L\sin(pt+\alpha)+M\sin(qt+\beta)\\y&=L'\sin(pt+\alpha)+M'\sin(qt+\beta)\end{aligned}\right\}\quad\text{.......................}\quad(2),$$

we substitute these in the right-hand sides of (1). The equations take the form

$$\left.\begin{aligned}(\delta^2-A)\,x-By&=\Sigma P\sin(\lambda t+\mu)\\-B'x+(\delta^2-C)\,y&=\Sigma Q\sin(\lambda t+\mu)\end{aligned}\right\}\quad\text{.......................}\quad(3),$$

where λ may have any one of the values 0, $2p$, $2q$, $p\pm q$ and P, Q contain the squares of the small quantities L, M, L', M'. To solve these equations, we consider only the specimen term of (3) and assume

$$\left.\begin{aligned}x&=L\sin(pt+\alpha)+M\sin(qt+\beta)+R\sin(\lambda t+\mu)\\y&=L'\sin(pt+\alpha)+M'\sin(qt+\beta)+R'\sin(\lambda t+\mu)\end{aligned}\right\}\quad\text{...............}\quad(4).$$

We find by an easy substitution

$$R(\lambda^2+A)+BR'=-P,\qquad B'R+R'(\lambda^2+C)=-Q;$$

$$\therefore\ R=\frac{-P(\lambda^2+C)+QB}{(\lambda^2+A)(\lambda^2+C)-BB'},\qquad R'=\frac{PB'-Q(\lambda^2+A)}{(\lambda^2+A)(\lambda^2+C)-BB'}.$$

It appears that R, R' are very small quantities of the second order, except when λ is such that the common denominator is small, and in this case R, R' may become very great. The roots of the denominator are $\lambda^2=p^2$, $\lambda^2=q^2$, and the denominator is small when λ is nearly equal to either p or q. This requires either that one of the two frequencies p, q should be small or that one should be nearly double the other.

If for example p is nearly equal to $2q$ and the numerators of R, R', are not thereby made small, the terms defined by $\lambda=p-q$ and $\lambda=2q$ will considerably influence the motion, the other terms producing no perceptible effect. If $p=2q$ exactly the denominator is zero and both R, R' take infinite values. The dynamical meaning of the infinite term is that the expressions (2) do not represent the motion with sufficient accuracy (except initially) to be a first approximation. The corrections to these expressions are found to become infinite and if we desire a solution we must seek some other first approximation.

304. Oscillation about steady motion. *Ex.* 1. The constituents of a multiple star describe circles about their centre of gravity O with a uniform angular velocity n, the several bodies always keeping at the same distances from each other. A planet P, *of insignificant mass*, freely describes a circle of radius a, centre O, with the same angular velocity, under the attraction of the other bodies. It is required to find the oscillations of P when disturbed from this state of motion.

Let $r=a(1+x)$, $\theta=nt+y$ be the polar coordinates of the planet P at any time t. Let the work function in the revolving field of force be

$$U-U_0=A_0x+B_0y+\tfrac{1}{2}(Ax^2+2Bxy+Cy^2)+\&c.\text{...... ....}(1),$$

at all points in the neighbourhood of the circular motion. Since that motion is possible only in that part of the field in which the force tends to O and is equal to n^2a, it is clear that $A_0 = -a^2n^2$ and $B_0 = 0$.

Substituting the values of r, θ in the polar equations

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{dU}{a\,dx}, \qquad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{dU}{r\,dy} \qquad (2),$$

we find the linear equations

$$\left.\begin{aligned}(a^2\delta^2 - a^2n^2 - A)\,x - (2a^2n\delta + B)\,y = 0\\ (2a^2n\delta - B)\,x + (a^2\delta^2 - C)\,y = 0\end{aligned}\right\} \qquad (3).$$

A principal oscillation is therefore given by

$$x = L\cos pt + L'\sin pt, \qquad y = M\cos pt + M'\sin pt \qquad (4),$$

$$M = \frac{2a^2npL' - BL}{a^2p^2 + C}, \qquad M' = \frac{-2a^2npL - BL'}{a^2p^2 + C} \qquad (5),$$

$$(a^2p^2 + A + a^2n^2)(a^2p^2 + C) - B^2 - 4a^4n^2p^2 = 0 \qquad (6).$$

The path of the particle when describing a principal oscillation relatively to its undisturbed path is the conic

$$(a^2p^2 + A + a^2n^2)\,x^2 + 2Bxy + (a^2p^2 + C)\,y^2 = \frac{4a^2n^2p^2}{a^2p^2 + C}(L^2 + L'^2) \qquad (7),$$

the ratio and directions of the axes being independent of the disturbance. In the limiting case in which $n = 0$ the conic reduces to two straight lines.

When the multiple star has *two constituents* A, B, whose masses are M, M', the planet P can describe a circular orbit only when $M\rho^{-\kappa}\sin APO = M'\rho'^{-\kappa}\sin BPO$, where $\rho = AP$, $\rho' = BP$ and the law of force is the inverse κth power of the distance. Since O is the centre of gravity of M, M' this proves that either the angle APO is zero or $\rho = \rho'$, except when $\kappa = -1$. The planet P must therefore be either in the straight line AB or at the corner C of the equilateral triangle ABC.

When the planet P is in the straight line AB at a point C such that the sum of the attractions of A and B on it is equal to $n^2 \cdot OC$, the planet can describe a circle about O with the same periodic time as A and B. This motion is unstable.

When the planet P is at the third corner C of the equilateral triangle ABC, the circular motion is stable when $\dfrac{(M+M')^2}{MM'} > 3\left(\dfrac{1+\kappa}{3-\kappa}\right)^2$.

These two results may be obtained in several ways. Putting ρ, ρ' for the distances of P from the two primaries the work function is

$$U = \frac{1}{\kappa - 1}\left(\frac{M}{\rho^{\kappa-1}} + \frac{M'}{\rho'^{\kappa-1}}\right).$$

Expressing this in terms of r, θ, and expanding in powers of x, y, including the terms of the second order, the values of A, B, C in equation (1) become known. The periods are then given by (6).

Instead of using the work function, we may determine the forces $dU/a\,dx$ and $dU/r\,dy$ by resolving the attractions of the primaries along and perpendicular to the radius vector of P. *This method has the advantage that the task of calculating the terms of the second order becomes unnecessary.*

Lastly, we may use the Cartesian equations referred to moving axes which rotate round O with a uniform angular velocity n, OC being the axis of ξ; Art. 227.

In all these methods, the assumption that the mass of the planet P is insignificant compared with that of either of the attracting bodies greatly simplifies the analysis. It does not seem necessary to examine these cases more fully here, as the results and the method of proceeding when this assumption is not made will be considered further on.

Ex. 2. If in the last example the attracting primaries either coincide or are so arranged that the field of force is represented by $U-U_0=A_0x+\frac{1}{2}Ax^2$; prove that other circular orbits in the immediate neighbourhood of the given one are possible paths for the particle P, Art. 291. Prove also that after disturbance the oscillation of P about the *mean circular path* is given by

$$x=L\cos(pt+\alpha), \qquad py=-2nL\sin(pt+\alpha),$$

where $p^2=3n^2-A/a^2$, the oscillation having only one period.

Ex. 3. Two equal centres of force S, S', whose attraction is $\mu\rho^\kappa$, rotate round the middle point O of the line of junction with a uniform angular velocity n. A particle in equilibrium at O is slightly disturbed, prove that the periods of the small oscillation are given by $(p^2+n^2-\beta)(p^2+n^2-\kappa\beta)=4n^2p^2$ where $\beta=2\mu b^{\kappa-1}$ and $SS'=2b$. Thence deduce the conditions that the equilibrium should be stable.

Problems requiring Finite Differences.

305. *Ex.* 1. A light elastic string of length nl and coefficient of elasticity E is loaded with n particles each of mass m, ranged at intervals l along it beginning at one extremity. If it be hung up by the other extremity, prove that the periods of its vertical oscillations will be given by the formula

$$\pi\sqrt{\frac{lm}{E}}\cdot\operatorname{cosec}\frac{2i+1}{2n+1}\frac{\pi}{2}, \text{ where } i=0,1,2\ldots n-1^*. \qquad \text{[Math. Tripos, 1871.]}$$

Let x_κ be the distance of the κth particle from the fixed end O; T_κ the tension above, $T_{\kappa+1}$ that below, the particle. We then have

$$mx_\kappa''=mg+T_{\kappa+1}-T_\kappa \quad\ldots\ldots(1),$$

and by Hooke's law for elastic strings

$$T_\kappa=E\left(\frac{x_\kappa-x_{\kappa-1}}{l}-1\right) \quad\ldots\ldots(2).$$

The equation of motion is therefore

$$x_\kappa''-g=c^2(x_{\kappa+1}-2x_\kappa+x_{\kappa-1}) \quad\ldots\ldots(3),$$

where $c^2=E/lm$. We assume as the trial solution

$$x_\kappa=h_\kappa+X_\kappa\sin(pt+\epsilon) \quad\ldots\ldots(4),$$

where h_κ and X_κ are two functions of κ which are independent of t, and p, ϵ are independent of both κ and t. Substituting we find

$$\left.\begin{aligned} X_{\kappa+1}-2X_\kappa+X_{\kappa-1}&=-\frac{p^2}{c^2}X_\kappa \\ h_{\kappa+1}-2h_\kappa+h_{\kappa-1}&=-\frac{1}{c^2}g \end{aligned}\right\} \quad\ldots\ldots(5).$$

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* The solution is given at greater length than is necessary for this example, in order to illustrate the various cases which may arise.

To solve the first of these linear equations of differences we follow the usual rules. Taking $X_\kappa = A a^\kappa$ as a trial solution, where A and a are two constants, we get after substitution and reduction

$$a - 2 + \frac{1}{a} = -\frac{p^2}{c^2} \quad\text{......(6)},$$

$$\therefore \sqrt{a} = \pm\sqrt{\left(1 - \frac{p^2}{4c^2}\right)} + \frac{p}{2c}\sqrt{-1} \quad\text{......(7)}.$$

Let these values of a be called α and β. Then

$$X_\kappa = A\alpha^\kappa + B\beta^\kappa \quad\text{......(8)}.$$

We notice that when either $p=0$ or $2c$ the equation (6) has *equal roots*, viz. $a=1$ or -1. The theory of linear equations shows that the terms depending on these values of p take a different form, viz.

$$X_\kappa = (A + B\kappa)(\pm 1)^\kappa \quad\text{......(9)}.$$

The complete value of x_κ may be written in the form

$$x_\kappa = h_\kappa + A_0 + B_0\kappa + (A_{2c} + B_{2c}\kappa)(-1)^\kappa \sin(2ct + \epsilon_{2c})$$
$$+ \Sigma (A_p\alpha^\kappa + B_p\beta^\kappa)\sin(pt + \epsilon_p) \quad\text{......(10)},$$

where Σ implies summation for all existing values of p.

We have yet to examine the conditions at the extremities of the string. The formula (2) does not express the tension of the highest string unless we suppose that $x_0 = 0$. Again the tension below the lowest particle must be zero and this requires that $T_{n+1} = 0$. The equation (3) will therefore express the motion of every particle from $\kappa = 1$ to $\kappa = n$ only if we make

$$x_0 = 0, \quad x_{n+1} - x_n = l \quad\text{......(11)}.$$

Since $x_0 = 0$ *for all values of* t, it follows from (10) that

$$h_0 + A_0 = 0, \quad A_{2c} = 0, \quad A_p + B_p = 0 \quad\text{......(12)}.$$

Since $x_{n+1} - x_n = l$, we see in the same way that

$$h_{n+1} - h_n + B_0 = l, \quad B_{2c} = 0, \quad A_p\alpha^{n+1} + B_p\beta^{n+1} = A_p\alpha^n + B_p\beta^n \quad\text{......(13)}.$$

Eliminating the ratio A_p/B_p we have

$$\alpha^{n+1} - \beta^{n+1} = \alpha^n - \beta^n \quad\text{......(14)}.$$

If $p > 2c$ we see by (7) that both α and β are real negative quantities. The equation (14) has then one side positive and the other negative, since the integers n, $n+1$ cannot be both even or both odd. Hence p must be less than $2c$, let $p = 2c\sin\theta$, hence

$$\alpha = \cos 2\theta + \sin 2\theta\sqrt{-1}, \quad \beta = \cos 2\theta - \sin 2\theta\sqrt{-1} \quad\text{......(15)}.$$

The equation (14) now gives $\sin(2n+2)\theta = \sin 2n\theta$, excluding $p=0$ we have

$$\theta = \frac{2i+1}{2n+1}\frac{\pi}{2}, \qquad \frac{p}{2c} = \sin\frac{2i+1}{2n+1}\frac{\pi}{2} \quad\text{......(16)},$$

where i has any integer value. It is however only necessary to include the values $i=0$ to $i=n-1$. The values of θ indicated by $i=i'$ and $2n-i'$ are supplementary, while the values of $\sin\theta$ indicated by $i=i'$ and $i'+2n+1$ are equal with opposite signs. The value $i=n$ is excluded because the value $p=2c$ has been already taken account of.

The oscillations of the κth particle are therefore given by

$$x_\kappa = H_\kappa + \Sigma C_p \sin 2\kappa\theta \sin(pt + \epsilon_p) \quad\text{......(17)},$$

where $H_\kappa = h_\kappa + A_0 + B_0\kappa$, and $C_p = 2A_p\sqrt{-1}$.

The value of h_κ might be determined by solving the second equation of differences (5), using the rules of linear equations adapted to that equation. But it is evident that in the position of equilibrium of the system, when there is no oscillation, every $C_p=0$, and therefore that position is determined by $x_\kappa=H_\kappa$. This enables us to deduce H_κ from the elementary rules of Statics.

We notice that in equilibrium, $T_n=mg$, $T_{n-1}=2mg$, &c., $T_\kappa=(n+1-\kappa)\,mg$. Hence by Hooke's law

$$H_\kappa - H_{\kappa-1} = l + (n+1-\kappa)\,g/c^2.$$

Adding these for all values of κ from $\kappa=1$ to $\kappa=\kappa$, and remembering that $H_0=0$ by (12), we find

$$H_\kappa = \left\{l + \frac{2n+1}{2}\,\frac{g}{c^2}\right\}\kappa - \frac{g}{2c^2}\kappa^2 \quad\text{.........................}\quad (18).$$

The equation (17) shows that the motion of every particle is compounded of n principal or simple harmonic oscillations. The periods of these are unequal and are represented by $2\pi/p$ where p has the values given in (16).

Suppose the system to be performing the principal oscillation defined by the value of $\theta=\pi/2\gamma$. By considering the signs of $\sin 2\kappa\theta$ in (17) we see that all the particles determined by $\kappa<\gamma$ are moving in the same direction as the highest particle, those determined by $\kappa>\gamma$ but $<2\gamma$ are moving in the opposite direction, those given by $\kappa>2\gamma$ but $<3\gamma$ are moving at any time in the same direction, and so on.

Ex. 2. A smooth circular cylinder is fixed with its axis horizontal at a height h above the edge of a table. A light string has a series of particles attached to it over a part of its length, the particles being each of mass m and distant a apart. The portion of the string to which the particles are attached is coiled up on the table, and the rest is carried over the cylinder, and a mass M attached to the further end of it. The system is held so that the first particle is just in contact with the table, the free portions of the string being vertical, and is then allowed to move from rest; prove that if v be the velocity of the system immediately after the nth particle is dragged into motion ($na<h$), then

$$v^2 = \frac{(n-1)\,ga}{3}\cdot\frac{6M^2 - n(2n-1)\,m^2}{(M+nm)^2}.$$

Supposing the string of particles to be replaced by a uniform chain deduce from the above result the velocity of the system after a length x of the chain has been dragged into motion. If l be the length of the chain and μ the mass, then, if l be less than h, the amount of energy that will have been dissipated by the time the chain leaves the table will be $\dfrac{\mu g l}{6}\,\dfrac{3M-\mu}{M+\mu}$. [Coll. Ex. 1887.]

If v_n represent the velocity required, we deduce from vis viva and linear momentum at the next impact the equation

$$\{M+(n+1)\,m\}^2 v_{n+1}^2 - \{M+nm\}^2 v_n^2 = 2ga\,(M^2 - n^2m^2).$$

Writing the left-hand side $\phi(n+1)-\phi(n)$, we find $\phi(n+1)-\phi(1)$ by summing from $n=1$ to n. Remembering that $v_1=0$, this gives v_n. The energy dissipated is found by subtracting the semi vis viva, viz. $\frac{1}{2}(M+\mu)\,v^2$, from the work done by gravity, viz. $(M-\frac{1}{2}\mu)\,lg$.

Ex. 3. A train of an engine and n carriages running with a velocity u, is brought to rest by applying the brakes to the engine alone, the steam being cut off. There is a succession of impacts between the buffers of each carriage and the next following. Prove that the velocity v of the engine immediately after the rth impact is given by

$$(M+rm)^2(v-u)^2=Mafr\{2M+m(r-1)\},$$

where m is the mass of any carriage, M that of the engine, a the distance between the successive buffers when the coupling chains are tight, f the retardation the brake would produce in the engine alone. [Coll. Ex.]

Ex. 4. A heavy particle falls from rest at a given altitude h in a medium whose resistance varies as the square of the velocity. On arriving at the ground it is immediately reflected upwards with a coefficient of elasticity β. Show that the whole space described from the initial position to the ground at the nth impact is $\dfrac{L^2}{g}\log\left\{1+\dfrac{1-\beta^{2n}}{1-\beta^2}\left(e^{\frac{2gh}{L^2}}-1\right)\right\}-h.$

If u_n be the height described just after the nth rebound, we show

$$u_n(u_{n+1}-1)=\beta^2(u_n-1).$$

To solve this equation of differences we put $u_n=1+1/w_n$. The equation then takes a standard form with constant coefficients. The whole space described is found by taking the logarithm of the product $u_0u_1u_2\ldots u_{n-1}$.

This problem was first solved by Euler in his *Mechanica*, vol. I. prop. 58, for the case in which $\beta=1$. An extension by Dordoni, *Memorie della Societa Italiana*, 1816, page 162, is mentioned in Walton's *Mechanical Problems*, chap. II. page 247.

CHAPTER VI.

CENTRAL FORCES.

Elementary Theorems.

306. *To find the polar equations of motion of a particle describing an orbit about a centre of force.*

Let the plane of the motion be the plane of reference and let the origin be at the centre of force. Let F be the accelerating force at any point measured positively towards the origin. Then by Art. 35,

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -F, \quad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0 \text{..........(1).}$$

The latter equation gives by integration

$$r^2 d\theta/dt = h \text{..........................(2),}$$

where h is an arbitrary constant whose value depends on the initial conditions.

This important equation can be put into other forms of which much use is made. Let v be the velocity of the particle, p the perpendicular drawn from the origin on the tangent. Let A be the area described by the polar radius as it moves from some initial position to that which it has at the time t. Then (Art. 7)

$$r^2 d\theta = 2dA = pds.$$

Remembering that $v = ds/dt$, we see that the equation (2) may be written in either of the forms

$$v = \frac{h}{p}, \qquad \frac{dA}{dt} = \tfrac{1}{2}h \text{.......................(3).}$$

The first of these shows that *the velocity at any point of the orbit is inversely proportional to the perpendicular drawn from the centre on the tangent.* The second, by integration between the limits $t = t_0$ to t, shows that *the polar area traced out by the radius vector*

is proportional to the time of describing it. We also see that the constant h represents twice the polar area described in a unit of time. Both these are Newtonian theorems.

We also infer that in a central orbit, the angular velocity $d\theta/dt$ always keeps one sign and never vanishes at a finite distance from the origin. The radius vector therefore continually turns round the origin in the same direction.

307. Conversely, we may show that if a particle so move that the radius vector drawn from the origin describes areas proportional to the time the resultant force always tends to the origin and is therefore a central force. To prove this let F and G be the components of the accelerating force along and perpendicular to the radius vector. Taking the transversal resolution, we have

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = G.$$

As already explained $r^2 d\theta = 2dA$, and if the area A bear a constant ratio to the time, say $A = at$, we have at once $r^2 d\theta/dt = 2a$ and therefore $G = 0$.

308. If m is the mass of the particle, its linear momentum is mv and this being directed along the tangent to the path, the moment of the momentum about the centre of force is $mv \cdot p$. The moment of the momentum is called *the angular momentum* (Art. 79) and we see that *in a central orbit the angular momentum about the centre of force is constant and equal to mh.* When we are concerned only with a single particle its mass is usually taken to be unity, and h then represents the angular momentum.

309. *To find the polar equation of the orbit* we must eliminate t from the equations (1). Let $r = 1/u$, then, as in Art. 268,

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta},$$

$$\frac{d^2r}{dt^2} = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2}.$$

Substituting this value of d^2r/dt^2 and the value of $d\theta/dt = hu^2$ given by (2) in $\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -F$, we have

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2} \quad\text{..........................(4).}$$

When the polar equation of the path is given in the form $u=f(\theta)$ the equation (4) determines F in terms of u and θ. Since the attractive forces of the bodies which form the solar system are in general functions of the distance only we should eliminate θ by using the known polar equation of the path. We thus find F as a function of u only.

Strictly this expression for F only holds for points situated on the given path, but *if the initial conditions are arbitrary*, the path may be varied and the law of force may be extended to hold for other parts of space.

When the force F is given as a function of r or $1/u$, the equation (4) is a differential equation of the form $\dfrac{d^2u}{d\theta^2}=f(u)$. This differential equation has been already solved in Art. 97.

It is evident from dynamical considerations that when the central force is attractive, i.e. when F is positive, the orbit must be concave to the centre of force, and when F is negative the orbit must be convex. By looking at equation (4) we immediately verify the theorem in the differential calculus that a curve is concave or convex to the origin according as $\dfrac{d^2u}{d\theta^2}+u$ is positive or negative.

310. *To apply the tangential and normal resolutions to a central orbit.*

Referring to Art. 36 we have the two equations

$$v\frac{dv}{ds}=-F\cos\phi, \quad \frac{v^2}{\rho}=F\sin\phi \;\ldots\ldots\ldots\ldots\ldots(5),$$

where ϕ is the angle behind the radius vector when the particle moves in the direction in which s is measured. Writing dr/ds for $\cos\phi$ and integrating we have

$$v^2=C-2\int F\,dr\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

where C is a constant whose value depends on the initial conditions. This equation is obviously the equation of vis viva, Art. 246. The integral has a minus sign because the central force is, as usual, measured positively towards the origin, while the radius vector is measured positively from the origin.

If we substitute for v its value h/p given by (3) and differentiate we deduce

$$F = -\tfrac{1}{2}h^2 \frac{d}{dr}\left(\frac{1}{p^2}\right) \quad\text{.....................(7).}$$

This expression for the central force F is very useful when the orbit is given in the form $p = f(r)$.

311. Considering the normal resolution (5), we have an expression for v which is useful when both the law of force and the path are known. It has the advantage of giving the velocity without requiring the previous determination of either of the constants C or h. If χ is one-quarter of the chord of curvature of the path drawn in the direction of the centre of force we may write the equation in either of the forms

$$v^2 = F\rho \sin\phi = 2F\chi \quad\text{.............. ......(8).}$$

This is usually read; *the velocity at any point is that due to one-quarter of the chord of curvature.*

When the particle describes a circle about a centre of force in the centre $\sin\phi = 1$ and ρ is the radius r. The velocity given by the normal resolution, viz. $v^2/r = F$, is often called *the velocity in a circle at a distance r from the centre of force.*

312. The velocity acquired by a particle which travels from rest at an infinite distance from the centre of force to any given position P is called *the velocity from infinity.* Referring to the equation of vis viva (6), let

$$F = \frac{\mu}{r^n}; \quad \therefore\ v^2 = C + \frac{2\mu}{n-1}\frac{1}{r^{n-1}}.$$

Now $v = 0$ when $r = \infty$; hence, if n is greater than unity, we have $C = 0$. The velocity from infinity to the distance $r = R$ is therefore given by $v^2 = \dfrac{2\mu}{n-1}\dfrac{1}{R^{n-1}}$. See Art. 181.

If n is less than unity the value of C is infinite. Instead of the velocity from infinity we use *the velocity acquired by the particle in travelling from rest at the given point P to the origin* under the attraction of the central force. In this case $v = 0$ when $r = R$; hence (since $n < 1$) $C = \dfrac{2\mu}{1-n}R^{1-n}$. The velocity to the origin (where $r = 0$) is then given by $v^2 = \dfrac{2\mu}{1-n}R^{1-n}$.

When the force varies as the inverse cube of the distance, i.e. $F=\mu/r^3$, we notice that the velocity in a circle and the velocity from infinity are equal. When the force varies as the distance, i.e. $F=\mu r$, the velocity in a circle is equal to that to the origin. When the force varies inversely as the distance, i.e. $F=\mu/r$, both the velocity from infinity and the velocity to the origin are infinite.

313. The constants. The two constants h and C may be determined from the initial conditions when these are known. Let the particle be projected from a point P at an initial distance R from the origin with a velocity V, let β be the angle the direction of projection makes with the initial radius vector. The tangent at P makes two angles with the radius vector OP, respectively equal to β and $\pi-\beta$. When a distinction has to be made it is usual to take β equal to the angle *behind the radius vector* when P travels along the curve in the positive direction (i.e. the direction which makes the independent variable increase). The angle β is called *the angle of projection.* We evidently have $h=vp=VR\sin\beta$. If $F=\mu/r^n$, we have $v^2=C+\dfrac{2\mu}{n-1}\dfrac{1}{r^{n-1}}$.

It follows that, if $n>1$ and the velocity from infinity is V_1, $C=V^2-V_1^2$; if $n<1$, $C=V^2+V_0^2$ where V_0 is the velocity to the origin.

We may obtain another interpretation for the constant C. Selecting any standard distance $r=a$, the potential energy at a distance r is

$$\int_r^a(-F)\,dr=\frac{\mu}{n-1}\left(\frac{1}{a^{n-1}}-\frac{1}{r^{n-1}}\right)=\frac{\mu}{(n-1)\,a^{n-1}}+\frac{C}{2}-\frac{v^2}{2}.$$

See Art. 250. It follows that $\frac{1}{2}C$ plus $\dfrac{\mu}{n-1}\dfrac{1}{a^{n-1}}$ is equal to the whole energy of the motion. *Hence by taking the standard position at infinity or the origin according as n is greater or less than unity, we may make $\frac{1}{2}C$ equal to the whole energy.*

314. When a point P on the orbit is such that the radius vector OP is perpendicular to the tangent, the point P is called *an apse.*

When OP is a maximum the apse is sometimes called *an apocentre,* and when a minimum *a pericentre.*

315. Summary. As the formulæ we have arrived at are the fundamental ones in the theory of central forces, it is useful to make a short summary before proceeding further. There are three elements to be considered: (1) the law of force, (2) the equations of the path, (3) the velocity and time of describing an arc. Any one of these elements being given, the other two can be deduced by dynamical considerations. There are therefore three sets of equations; firstly, equations (4) and (7) connect the force and path, so that either being known the other can be deduced; secondly, equation (6) connects the force and velocity; thirdly, equations (2) and (3) connect the path with the motion in that path.

The equations of one of these sets are mere algebraic transformations of each other, any one being given the others can be found from it by reasoning which is purely mathematical. But an equation of one set cannot be deduced from an equation of another set in this manner, because each set depends on different dynamical facts.

316. Dimensions. It is important to notice the dimensions of the various symbols used. The accelerating force F, like that of gravity, i.e. g, is one dimension in space and -2 in time. We see this by examining any formula which contains F or g, say $s=\frac{1}{2}gt^2$ or $-F\cos\phi=d^2s/dt^2$. The force F will in general vary as some power of the distance from the centre of force, say $F=\mu/r^n$ where μ is a constant which measures the strength of the central force. The quantity $\mu=Fr^n$ is therefore $n+1$ dimensions in space and -2 in time. The velocity $v=ds/dt$ is one dimension in space and -1 in time. The constant $h=vp$ is 2 dimensions in space and -1 in time. See Art. 151.

317. Force given, find the orbit. *Ex.* 1. The force being

$$F=\mu u^3(2a^2u^2+1),$$

a particle is projected from an initial distance a, with a velocity which is to the velocity in a circle at the same distance as $\sqrt{2}$ to $\sqrt{3}$, the angle of projection being 45°. Find the path described.

Putting $a=1/c$ the differential equation of motion is, by Ar' `09,

$$h^2\left(\frac{d^2u}{d\theta^2}+u\right)=\frac{2\mu}{c^2}u^5+\mu u;$$

$$\therefore\ v^2=h^2\left\{\left(\frac{du}{d\theta}\right)^2+u^2\right\}=\frac{\mu}{c^2}u^4+\mu u^2+C.$$

When $u=c$, the conditions of the question give $v^2=\frac{2}{3}F/c$ and $h=v\sin\beta/c$ where $\sin^2\beta=\frac{1}{2}$, see Arts. 311, 313. We therefore have $C=0$, $h^2=\mu$. The equation now reduces to

$$\left(\frac{du}{d\theta}\right)^2=\frac{u^4}{c^2}; \qquad \therefore \int\frac{du}{u^2}=\pm\frac{\theta}{c}+A.$$

Replacing u by $1/r$ and measuring θ from the initial radius OA in such a direction that r and θ increase together, this leads to $r=a(1+\theta)$.

From the equation $r^2d\theta/dt=h$, we infer that the time from a distance a to r is $(r^3-a^3)/3a\sqrt{\mu}$.

Ex. 2. A particle moves under the action of a central force $\mu(u^5-\frac{1}{8}a^2u^7)$, the velocity of projection being $(25\mu/8a^4)^{\frac{1}{2}}$, and the angle of projection $\sin^{-1}\frac{4}{5}$. Prove that the polar equation of the path is $3a^2=(4r^2-a^2)(\theta+C)^2$. [Coll. Ex. 1892.]

Ex. 3. When the central acceleration is $\mu(u^3+a^2u^5)$ and the velocity at the apsidal distance a is equal to $\sqrt{\mu}/a$, prove that the orbit is $r=a\,\mathrm{cn}\,\theta$ (mod $\sqrt{\frac{1}{2}}$). [Coll. Ex. 1897.]

Ex. 4. The central force being $F=2\mu u^3(1-a^2u^2)$, the particle is projected from an apse at a distance a with a velocity $\sqrt{\mu}/a$. Prove that it will be at a distance r after a time $\frac{1}{2\sqrt{\mu}}\left\{a^2\log\frac{r+\sqrt{(r^2-a^2)}}{a}+r\sqrt{(r^2-a^2)}\right\}$. [Math. Tripos.]

Ex. 5. A particle, acted on by two centres of force both situated at the origin respectively $F=\mu u^3$ and $F'=\mu' u^5$, is projected from an initial distance a with a velocity equal to that from infinity, the angle of projection being $\tan^{-1}\sqrt{2}$. If the forces are equal at the point of projection, the path is $a\theta=(r-a)\sqrt{2}$.

Ex. 6. A particle, acted on by the central force $F=u^2f(\theta)$, is initially projected in any manner. Prove that the radius vector can be expressed as a function of θ if the integrals of $\cos\theta f(\theta)$ and $\sin\theta f(\theta)$ can be found. [Use the method of Art. 122.]

318. Orbit given, find the force. *Ex.* 1. *A particle describes a given circle about a centre of force on the circumference. It is required to find the law of force and the motion. Newton's problem.*

Let O be the centre of force, C the centre of the circle, P the particle at the time t. Let a be the radius of the circle, $OP=r$. If $p=OY$ be the perpendicular on the tangent, we have (since the angles OPY, OAP are equal) $p=r^2/2a$. Hence using (7) of Art. 310, we have

$$F=-\tfrac{1}{2}h^2\frac{d}{dr}\frac{1}{p^2}=\frac{8h^2a^2}{r^5}\quad\text{.......(1).}$$

If we suppose the magnitude of the force to be given at a unit of distance from the centre of force we write this in the form $F=\frac{\mu}{r^5}$, where μ is a known constant sometimes called the magnitude or strength of the force. The constant h is then determined by the equation

$$8h^2a^2=\mu\quad\text{.......(2).}$$

The velocity at any point P is found by the normal resolution, Art. 310,

$$\frac{v^2}{a}=F\sin OPY=\frac{\mu}{r^5}\frac{r}{2a}; \qquad \therefore v=\sqrt{\frac{\mu}{2}}\cdot\frac{1}{r^2}\quad\text{.......(3).}$$

By Art. 312 this velocity is equal to that from infinity.

To find the time of describing any arc AP, where A is the extremity of the diameter opposite to the centre of force, we use the equation $A = \frac{1}{2}ht$, Art. 306. Since the area AOP is made up of the triangle OCP and the sector ACP, we have

$$\tfrac{1}{2}ht = A = \tfrac{1}{2}a^2(2\theta + \sin 2\theta),$$

where θ = the angle AOP. Substituting for h

$$t = 2a^3\sqrt{\frac{2}{\mu}}\,(2\theta + \sin 2\theta) \qquad\qquad (4).$$

It appears from this that the particle will arrive at the centre of force after a finite time obtained by writing $\theta = \frac{1}{2}\pi$. The particle arrives with an infinite velocity due to the infinite force at that point.

Let the force at all points of space act towards the point O and vary as the inverse fifth power of the distance from O. It is required to find *the necessary and sufficient condition that a particle projected from a given point P in a given direction PT with a given velocity V may describe a circle passing through O.* It is obvious from (3) that it is necessary that $V^2 = \frac{1}{2}\mu/r^4$ where $r = OP$; we shall now prove that this is also sufficient.

Describe the circle which passes through O and touches PT at P. The particle which describes this circle freely satisfies the given conditions at P. If then the given particle does not also describe the circle we should have two particles projected from P in the same direction, with equal velocities, acted on by the same forces, describing different paths; which is impossible; Art. 243.

We notice that a change in the direction of projection PT affects the size of the circle described, but not the fact that the path is a circle.

Ex. 2. A particle moves in a circle about a centre of force in the circumference, the force being attractive and equal to μr^n. Prove that the resistance of the medium in which the particle moves is $\frac{1}{4}\mu\,(n+5)\,r^n \sin\theta$, where $\cos\theta = r/2a$.

Use the normal and tangential resolutions. [Coll. Ex.]

Ex. 3. A particle of unit mass describes a circle about a given centre of force O situated on the circumference. If the particle at any point P is acted on by an impulse $2v\cos\phi$ in a direction making an angle $\pi - \phi$ with the direction of motion PT, show that the new orbit is also a circle and prove that the ratio of the radii is $\cos 2\phi + \sin 2\phi \cot\theta$, where θ is the angle OPT.

Ex. 4. The force being $F = \mu u^5$, a particle when projected from a point P with an initial velocity V, equal to that from infinity, describes the circle $r = 2a\cos\theta$; investigate the path when the initial velocity is $V(1+\gamma)$, where γ is so small that its square can be neglected.

Proceeding as in Art. 317, we find

$$v^2 = h^2\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} = \frac{\mu}{2}u^4 + C.$$

The conditions of the question give

$$C = \frac{\mu}{2}\,\frac{c^4}{\cos^4\alpha}(2\gamma + \gamma^2), \qquad h^2 = \frac{\mu}{2}c^2(1+\gamma)^2,$$

where $c = 1/2a$ and $\theta = \alpha$ initially. Putting $u = c\sec\theta + c\eta$ and neglecting the squares of η and γ, we arrive at

$$\frac{\cos^2\theta}{\sin\theta}\,\frac{d\eta}{d\theta} + \frac{\cos^3\theta - 2\cos\theta}{\sin^2\theta}\,\eta = \frac{-\gamma}{\sin^2\theta} + \frac{\gamma}{\cos^4\alpha}\,\frac{\cos^4\theta}{\sin^2\theta}.$$

Each side being a perfect differential, we find

$$\frac{\cos^2\theta}{\sin\theta}\eta=\kappa+\gamma\cot\theta-\frac{\gamma}{\cos^4\alpha}(\cot\theta+\tfrac{3}{2}\theta+\tfrac{1}{2}\sin\theta\cos\theta),$$

and κ is determined from the condition that $\eta=0$ when $\theta=\alpha$;

$$\therefore\ \kappa=-\gamma\cot\alpha+\frac{\gamma}{\cos^4\alpha}(\cot\alpha+\tfrac{3}{2}\alpha+\tfrac{1}{2}\sin\alpha\cos\alpha).$$

Putting $u=1/r$, we have $r=2a\cos\theta\,(1-\eta\cos\theta)$,

$$\therefore\ \frac{r}{2a}=\cos\theta-\kappa\sin\theta-\gamma\cos\theta+\frac{\gamma}{\cos^4\alpha}(\cos\theta+\tfrac{3}{2}\theta\sin\theta+\tfrac{1}{2}\sin^2\theta\cos\theta).$$

It has been assumed that $\cos\alpha$ is not small, the point P must therefore not be close to the centre of force. It easily follows that when

$$\theta=\tfrac{1}{2}\pi-\kappa+\tfrac{3}{4}\pi\gamma\sec^4\alpha,$$

the distance of the particle from the centre of force is of the order of small quantities neglected above.

Ex. 5. Any number of particles are projected in all directions from a given point P each with the velocity from infinity, the central force being $F=\mu u^5$. Prove that their locus at any instant is (θ being measured from OP)

$$\frac{(r^2+c^2-2cr\cos\theta)^{\frac{3}{2}}}{\sin^3\theta}\left\{\theta-\sin\theta\frac{(r^2+c^2)\cos\theta-2cr}{r^2+c^2-2cr\cos\theta}\right\}=A,$$

where $OP=c$ and A is a constant depending on the time elapsed.

319. *Ex.* 1. *A particle describes an equiangular spiral of angle α under the action of a centre of force in the pole, prove that*

$$F=\frac{\mu}{r^3},\qquad h=\sin\alpha\sqrt{\mu},\qquad v=\frac{\sqrt{\mu}}{r},\qquad 2\cos\alpha t\sqrt{\mu}=r_1^2-r_0^2,$$

where t is the time of describing the arc bounded by the radii vectores r_0, r_1. Conversely, a particle being projected from any point in any direction will describe an equiangular spiral about a centre of force whose law is $F=\mu/r^3$, provided the velocity of projection is $\sqrt{\mu}/r$, i.e. is equal to that from infinity.

Assuming $p=r\sin\alpha$ we follow the same line of reasoning as in Ex. 1 of Art. 318.

Ex. 2. A particle acted on by a central force moves in a medium in which the resistance is $\kappa(\text{vel.})^2$, and describes an equiangular spiral, the pole being the centre of force. Prove that the central force varies as $\frac{1}{r^3}e^{-2\kappa r\sec\alpha}$, where α is the angle of the spiral. [Math. Tripos, 1860.]

320. *Ex. A particle describes the curve $r^m=a\cos n\theta+b\sin n\theta$, under the action of a centre of force in the origin. Prove that*

$$F=\frac{\mu}{r^{2m+3}}+\frac{\mu'}{r^3},\qquad v^2=\frac{\mu}{m+1}\frac{1}{r^{2m+2}}+\frac{\mu'}{r^2}.$$

We notice (1) that the exponents of r are independent of n, (2) that, when $m+1$ is positive, the velocity at any point is that due to infinity. Art. 312.

Supposing the law of force and the velocity of projection to be given by these formulæ, let the particle be projected from any point P in any direction PT. The

four constants h^2, n, a, b are determined by

$$h^2(m+1)\left(\frac{n}{m}\right)^2(a^2+b^2)=\mu, \qquad h^2\left(1-\frac{n^2}{m^2}\right)=\mu',$$

joined to the conditions that the curve must pass through P and touch PT.

We find that n^2 and $\frac{\mu}{m+1}-\mu' R^{2m}\cot^2\phi$ have the same sign, where $R=OP$ and ϕ is the angle of projection. When the sign of n^2 thus determined becomes negative or zero the curve obviously changes into

$$r^m = a'e^{n\theta}+b'e^{-n\theta}, \text{ or } r^m = a+b''\theta,$$

where $4a'b'=a^2-b^2$ and b'' is the limit of bn when b is infinite and n zero.

It is useful to notice the following geometrical properties of the curve. If p be the perpendicular on the tangent, ϕ the angle the radius vector makes with the tangent

$$\tan\phi = -\frac{m}{n}\cot n\theta, \qquad \frac{1}{p^2}=\frac{n^2}{m^2}\frac{a^2+b^2}{r^{2m+2}}+\frac{m^2-n^2}{m^2}\frac{1}{r^2}.$$

This example includes many interesting cases. Putting $m=2$, $n=2$, we see that the lemniscate of Bernoulli could be described about a centre of force in the node varying as the inverse seventh power of the distance. Putting $m=n$, we have the path when the force varies as the inverse $(2m+3)$th power and the velocity is that from infinity. Writing $m=\frac{1}{2}$, $n=\frac{1}{2}$, we find the path is a cardioid when the central force varies as the inverse fourth power and the velocity is that from infinity. Writing $m=1$, $n=1$, the path is a circle described about a centre of force on the circumference.

321. *Ex.* 1. *A particle describes a circle about a centre of force situated in its plane. It is required to find the law of force and the motion.*

Let O be the centre of force, C the centre of the circle, a its radius and $CO=c$. Taking the equations of Art. 310, we have

$$v=\frac{h}{p}, \qquad \frac{v^2}{a}=F\frac{p}{r}, \qquad \therefore\ F=\frac{h^2}{a}\frac{r}{p^3}.$$

Since in a circle $2ap=r^2+a^2-c^2$, we can, by substitution, express F and v in terms of r alone. We have

$$v=\left(\frac{\mu}{2}\right)^{\frac{1}{2}}\frac{1}{r^2+B}, \qquad F=\frac{\mu r}{(r^2+B)^3},$$

where $8a^2h^2=\mu$ and $B=a^2-c^2$. When $B=0$, the law of force reduces to the inverse fifth power, and the velocity becomes the same as that found in Art. 318.

If this law of force be supposed to hold throughout the plane of the circle, the values of μ and B are given. In order that the orbit may be a circle it is necessary that the velocity of projection should satisfy the above value of v, i.e. should be equal to the velocity from infinity. The direction of projection being also given, the angular momentum h (Art. 313) is also known. The values of a and c follow at once from the equations given above and must be real.

Newton, when discussing this problem, supposes that the centre of force lies inside the circle. It follows that B is positive, and at no point of space can either the force or velocity be infinite.

When the centre of force is outside the circle, one portion of the orbit is concave and the other convex to the centre of force. We must therefore suppose

that the force is attractive in the first and repulsive in the other part. Writing $B=-b^2$, we have $b^2=c^2-a^2$, and therefore b is the length of either of the tangents drawn from the centre of force to the circle, and the force changes sign through infinity when the particle passes the circle whose radius is b.

Sylvester, in the *Phil. Mag.* 1865, points out that the resultant attraction of a circular plate, whose elements attract according to the law of the inverse fifth power, at an external point P situated in its plane, is $\frac{\mu r}{(r^2-b^2)^3}$ where μ is the mass of the plate, b its radius and r the distance of P from the centre. The circle described by P under the attraction of this plate cuts the rim orthogonally.

Let the particle P be constrained to move on a smooth plane under the action of a centre of force situated at a point C distant b' from the plane, the law of force being the inverse fifth power. The component of force in the plane is $F=\frac{\mu r}{(r^2+b'^2)^3}$, where r is the distance of P from the projection O of the centre of force on the plane. Putting $B=b'^2$, it appears from what precedes that, if the velocity of projection is equal to that from infinity, the path of the particle on the plane is a circle. The length of the chord bisected by the point O is constant for all the circles and equal to $2b'$.

Ex. 2. A particle moves under the action of a centre of force $F=\mu u^5$. Prove that all the circles which can be described either pass through a fixed point or have a fixed point for centre.

322. *Ex.* 1. *A particle moves under the action of a centre of force whose attraction is* $F=\frac{\mu r}{(r^2+B)^2}$ *and the velocity at any point is equal to that from infinity. It is required to find the path.*

The equation of vis viva (Art. 310) gives

$$v^2=C-2\int F\,dr=C+\frac{\mu}{r^2+B} \quad\text{.............................(1).}$$

Since this formula is independent of the path and it is given that v is zero when r is infinite we see that $C=0$. Substituting for v its value h/p, the equation of the path becomes

$$r^2+B=ip^2, \qquad ih^2=\mu \quad\text{................................(2).}$$

The curve required is therefore such that a linear relation exists between p^2 and r^2. There are several species of curves which possess this property distinguished from each other by the values of B and i.

One such curve is known to be an epicycloid. Supposing the radii of the fixed and rolling circles to be a and b, we have at the cusp $r=a$, $p=0$ and at the vertex p and r are each equal to $a+2b$. We thus find

$$B=-a^2, \qquad \frac{\mu}{h^2}=i=\frac{(a+2b)^2-a^2}{(a+2b)^2} \quad\text{...........................(3).}$$

The law of force and the conditions of projection being given both B and h^2 are known. If the force is attractive, B negative, and μ/h^2 less than unity, the path is an epicycloid, the values of a and b being given by (3).

Changing the sign of b the epicycloid becomes a hypocycloid and in this case we learn from (3) that i and μ are negative. When therefore the force is repulsive, and B negative, the path is a hypocycloid.

The remaining species are more easily separated by putting the equation (2) into the form $\rho = ip$, a result which follows at once from the identity $\rho = r\,dr/dp$. Remembering that $\rho = p + d^2p/d\psi^2$ the differential equation becomes

$$\frac{d^2p}{d\psi^2} - (i-1)p = 0 \qquad\qquad (4).$$

When i is less than unity or is negative we easily deduce the cycloidal species given above. If $\beta^2 = 1 - i$, we find

$$p = L \sin \beta\psi + M \cos \beta\psi.$$

If the axis of x pass through the cusp, we have $p = 0$ when $\psi = 0$ and $p = a + 2b$ when $\beta\psi = \frac{1}{2}\pi$. Hence $L = a + 2b$ and $M = 0$.

When i is greater than unity we have the forms

$$p = Le^{\alpha\psi} + Me^{-\alpha\psi}, \qquad p = L\psi + M \qquad\qquad (5),$$

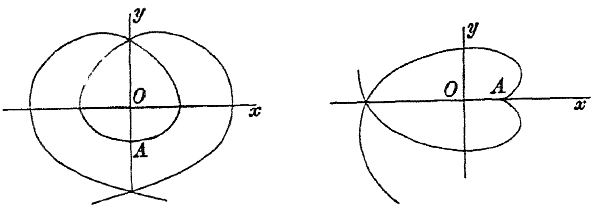
where $\alpha^2 = i - 1$ and the second form occurs when $i = 1$. Since in any curve the projection of the radius vector on the tangent is $dp/d\psi$, we find by elementary geometry

$$r^2 = p^2 + \left(\frac{dp}{d\psi}\right)^2, \qquad \tan\phi = \frac{p\,d\psi}{dp} \qquad\qquad (6),$$

where ϕ is the angle behind the radius vector. Since $\phi = \psi - \theta$, we can in this way express the polar coordinates r and θ in terms of the subsidiary angle ψ.

Substituting in (2) we find that $4\alpha^2LM = B$, so that L and M have the same or opposite signs according as the given quantity B is positive or negative. When $B = 0$, either L or M is zero, and since, by (6), $\tan\phi$ is then constant the curve is an equiangular spiral.

To trace the forms of the exponential spirals it is convenient to turn the axis



of x round the origin so that the equation (5) may assume a symmetrical form. We then have

$$p = \tfrac{1}{2}c\,(e^{\alpha\psi} \pm e^{-\alpha\psi}) \qquad\qquad (7),$$

where the upper or lower sign is to be taken according as B is positive or negative. When B is positive there is an apse whose position is found by putting $p = r$ in (2), whence $(i-1)\,r^2 = B$. When B is negative there is a cusp at the point determined by $p = 0$, i.e. at $r^2 = -B$. These spirals were first discussed by Puisseux (with a different object in view) in *Liouville's Journal*, 1844.

By using a proposition in the theory of attractions we may put some of the preceding problems in another light. It may be shown that the resultant attraction of a thin circular ring, whose elements attract according to the law of the inverse cube, at any point P in the plane of the ring is $\dfrac{\pm \mu r}{(r^2 - c^2)^2}$, where μ is the mass of the ring, c its radius and r the distance of P from the centre. The plus or minus

sign is to be taken according as P is without or within the ring, (see Townsend in the *Quarterly Journal*, 1879). The path of the particle P moving under the attraction of the ring has now been found provided the velocity of projection is equal to that from infinity.

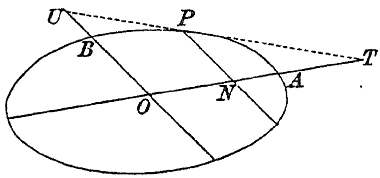
Again, when a particle P is constrained to move on a smooth plane under the action of a centre of force C situated at a distance c from the plane, the law of force being the inverse cube, the component of attraction in the plane is $\frac{\mu r}{(r^2+c^2)^2}$, where r is the distance of P from the projection O of the centre of force on the plane.

Ex. 2. If s be the arc AP of any path measured from a fixed point A, show that $s(i-1)/i$ differs from the projection of the radius vector OP on the tangent at P by a constant quantity which is zero when A is an apse.

Ex. 3. Show that the polar area traced out by a radius vector OP is equal to i times the corresponding polar area of the pedal. Thence show that the time of describing any arc is given by $ht=i\int p^2 d\psi$.

323. Parallel forces. *Ex.* 1. *A particle describes a central conic under the action of a force F tending always in a fixed direction. It is required to find F.*

Let the conic be referred to conjugate diameters OA, OB; the force acting



parallel to BO. Let the angle $AOB=\omega$, $OA=a'$, $OB=b'$. Let $ON=x$, $PN=y$ be the coordinates of P. Then

$$d^2x/dt^2=0, \qquad d^2y/dt^2=-F.$$

The first equation gives $x=At$, where A is the *oblique* component of velocity parallel to x. Hence A is the resultant velocity at B. We then have

$$y=\frac{b'}{a'}(a'^2-x^2)^{\frac{1}{2}}, \qquad \therefore \frac{d^2y}{dt^2}=-\frac{b'^4A^2}{a'^2}\frac{1}{y^3}.$$

The component of velocity at right angles to the force is constant. Representing this component by V, and remembering that the resultant velocity at B is A, we find $V=A\sin\omega$.

If a, b are the semi-axes of the conic the expression for the force becomes

$$F=\frac{V^2b'^4}{a'^2\sin^2\omega}\frac{1}{y^3}=\frac{V^2b'^6}{a^2b^2}\frac{1}{y^3}.$$

It follows that *the force tending in a given direction by which a conic can be described varies inversely as the cube of the chord along which the force acts.* This result may also be obtained without difficulty by taking the normal resolution of force.

Ex. 2. If the tangent to the conic at P intersect the conjugate diameters in T and U, prove that the velocity at P is $v=Ax\,.\,TU/a'^2$.

Ex. 3. A particle describes the curve $y=f(x)$ freely under the action of a force F whose direction is parallel to the axis of y; prove $F=A^2d^2y/dx^2$.

Ex. 4. Show that a particle can describe a complete cycloid freely under the action of a force tending towards the straight line joining the cusps and varying inversely as the square of the distance. Prove also that the square of the velocity varies inversely as the distance.

324. *Ex.* Two masses M, m are connected by a string which passes through a hole in a smooth horizontal plane, the mass m hanging vertically. Prove that M describes on the plane a curve whose differential equation is

$$\left(1+\frac{m}{M}\right)\frac{d^2u}{d\theta^2}+u=\frac{mg}{M}\,\frac{1}{h^2u^2}.$$

Prove also that the tension of the string is $\dfrac{Mm}{M+m}(g+h^2u^3)$. [Coll. Exam.]

Law of the direct distance.

325. *A particle is acted on by a centre of force situated in the origin whose acceleration is $F=\mu r$ where r is the radius vector. It is required to find the possible orbits.*

Taking any Cartesian axes, we notice that the resolved parts of the force in these directions are μx and μy. The equations of motion are therefore

$$d^2x/dt^2=-\mu x, \qquad d^2y/dt^2=-\mu y \qquad\text{..............(1)}.$$

We observe that though the axes of coordinates are arbitrary, the equations (1) are independent; one containing only x, the other only y. We infer that the general principle enunciated for parabolic motion may also be applied here. *The circumstances of the motion parallel to any fixed direction are independent of those in other directions and may be deduced from the corresponding formulæ for rectilinear motion.*

Supposing that the force is attractive in the standard case, μ is positive and the solutions of (1) are

$$x=A\cos\sqrt{\mu}t+A'\sin\sqrt{\mu}t, \quad y=B\cos\sqrt{\mu}t+B'\sin\sqrt{\mu}t.$$

As there is nothing to prevent us from using oblique axes, let us take the initial radius vector as the axis of x and let the axis of y be parallel to the direction of initial motion. If R and V be the initial distance and velocity, we have when $t=0$,

$$x=R, \quad dx/dt=0; \quad y=0, \quad dy/dt=V.$$

These give $\quad R=A, \quad 0=A', \quad 0=B, \quad V=B'\sqrt{\mu}.$

The motion is therefore determined by

$$x = R\cos\sqrt{\mu}t, \quad y = R'\sin\sqrt{\mu}t,$$

where $V = R'\sqrt{\mu}$. Eliminating t, we obviously arrive at the equation of a conic having its centre at the centre of force and R, R' for semi-conjugate diameters.

If μ is positive, the centre of force is attractive and the orbit must be at every point concave to the origin. *The orbit is therefore an ellipse. If μ is negative*, the central force repels, *and the orbit*, being convex to the origin, *is a hyperbola*. Since the centre of the conic is always at the centre of force the orbit can be a parabola only when the centre of force is infinitely distant. If the force at the particle is then finite, the coefficient μ must be zero. The finite changes of r as the particle moves about do not affect the value of μr. The force on the particle is then constant in magnitude and fixed in direction.

When μ is negative, we put $\mu = -\mu'$. The solution of the differential equations then becomes

$$x = \tfrac{1}{2}R\left(e^{\sqrt{\mu'}t} + e^{-\sqrt{\mu'}t}\right), \quad y = \frac{i}{2}R'\left(e^{\sqrt{\mu'}t} - e^{-\sqrt{\mu'}t}\right),$$

where $V = iR'\sqrt{\mu'}$ and $i = \sqrt{-1}$. It is evident that iR' is real.

326. Since any point of the orbit may be taken as the point of projection, we deduce from the equation $V = \sqrt{\mu}R'$, that *the velocity v at any point P of the ellipse is given by $v = \sqrt{\mu}R'$ where R' is semi-conjugate of OP.* If r be the radius vector of the moving particle this equation may also be written $v^2 = \mu(a^2 + b^2 - r^2)$ where a and b are the semi-axes.

Since $vp = h$ and $pR' = ab$, we see that *the constant h is $h = \sqrt{\mu}ab$.*

If the principal diameters are taken as the axes of coordinates, we have $x = a\cos\phi$, $y = b\sin\phi$, where ϕ is the eccentric angle of the particle. It immediately follows that *the particle so moves that $\phi = \sqrt{\mu}t$.* When ϕ has increased by 2π the particle has made a complete circuit and returned to its former position. *The periodic time is therefore $2\pi/\sqrt{\mu}$.* It appears from this that *the periodic time is independent of all the conditions of projection and is the same for all ellipses. It depends solely on the strength μ of the central force.*

In general the time of describing any arc PP' is the difference of the eccentric angles at P and P' divided by $\sqrt{\mu}$.

When the orbit is a hyperbola we have

$$x = \tfrac{1}{2}a\,(e^{\phi'} + e^{-\phi'}), \quad y = \tfrac{1}{2}b\,(e^{\phi'} - e^{-\phi'}),$$

where ϕ' is an auxiliary angle. It immediately follows that $\phi' = \sqrt{\mu'}t$ where μ' is positive and equal to $-\mu$.

327. When the velocity V and angle β of projection as well as the initial distance R are given, the semi-axes a, b of the conic described may be deduced from the equations

$$a^2b^2 = \frac{h^2}{\mu} = \frac{V^2R^2\sin^2\beta}{\mu}, \qquad a^2 + b^2 = R^2 + \frac{V^2}{\mu}.$$

These give real values to a^2 and b^2. The angle θ which the major axis makes with the initial distance is given by

$$\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} = \frac{1}{R^2}; \quad \therefore \tan^2\theta = \frac{b^2}{a^2}\cdot\frac{a^2 - R^2}{R^2 - b^2}.$$

Since $V = \sqrt{\mu}R'$, it is evident that the problem of finding the particular conic described when R and V are given is the same as *the geometrical problem of constructing a conic when two semi-conjugate diameters R, R' are given in position and magnitude.* This useful construction is given in most books on geometrical conics.

328. Referring to the equations (1) of Art. 325 we see that the motion in an ellipse about a centre of force $F = \mu r$ is the resultant of two rectilinear harmonic oscillations along two arbitrary directions Ox, Oy represented by

$$X = -\mu x, \; Y = -\mu y.$$

The resultant of any number of rectilinear harmonic oscillations (performed in equal times) along arbitrary straight lines OA, OB, &c. may be found by resolving the displacements of each along two arbitrary axes and compounding the sums of the components. The resulting motion is therefore an elliptic motion with O for centre.

Ex. Investigate the conditions that the resultant of two rectilinear harmonic oscillations, of equal periods, whose directions make an angle θ, should be (1) a rectilinear, (2) a circular motion. Prove that in the first case their angles or phases must be equal; in the second their amplitudes must be equal and their phases differ by $\pi - \theta$. The radius is $a \sin\theta$.

329. *Ex.* 1. If OP, OQ are conjugate diameters of an ellipse, prove that the time from P to Q is one-quarter of the whole periodic time. This follows at once from the fact that the area POQ is one-quarter of the area of the ellipse.

Ex. 2. Prove that in a hyperbolic orbit the time from the extremity of the major axis to a point whose distance from that axis is equal to the minor axis is the same for all hyperbolas.

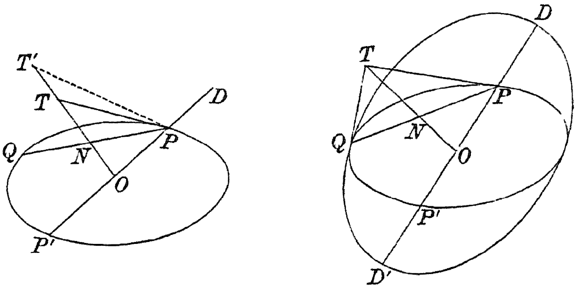
Ex. 3. If the circle of curvature at any point P of an ellipse cut the curve again in Q, and A is the extremity of the major axis nearest to P, prove that the time from Q to A is three times the time from A to P.

Since $\phi = \sqrt{\mu} t$, Art. 326, the theorems in conics which, like this one, are concerned with eccentric angles may at once be translated into dynamics.

Ex. 4. Two tangents TP, TQ are drawn to an ellipse, prove that the velocities at P and Q are proportional to the lengths of the tangents. [For these tangents are known to be proportional to the parallel diameters.]

330. Point to Point. *To find the directions in which a particle must be projected from a given point P with a given velocity V, so as to pass through another given point Q.*

Let r_1, r_2 be the distances of P, Q from the centre of force O. Let OP be produced to D where D is such that the velocity V of projection at P is equal to



that acquired by a particle starting from rest at D and moving to P under the action of the centre of force. Let $OD = k$. Then since $V^2 = \mu(a^2 + b^2 - r_1^2)$, the sum of the squares of any two semi-conjugates of the trajectory is k^2.

Bisect PQ in N and let $ON = x$, $NP = NQ = y$. From the equation of the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{k^2 - a^2} = 1;$$

$$\therefore \; a^4 - a^2(x^2 - y^2 + k^2) + k^2x^2 = 0 \quad \text{.........................(1).}$$

Since x, y, k are given, this quadratic gives two values of a^2, showing that there are two directions of projection which satisfy the given conditions.

Let these directions of projection from P intersect ON produced in T and T', then since $a^2 = ON \cdot OT$, the quadratic gives the positions of T and T'. We also have $OT \cdot OT' = k^2$, and $NT \cdot NT' = y^2$.

The roots of the quadratic (1) are imaginary if $x + y > k$. Produce PO to P' where $OP' = OP$, the roots of the quadratic are imaginary unless Q lie within the ellipse whose foci are P, P' and semi-major axis $a' = k$. *This ellipse is the boundary of all the positions of Q which can be reached by a particle projected from P with the given velocity.* It is also the envelope of all the trajectories.

Ex. 1. If two circles be described having their centres at O and N and their radii equal to k and y respectively, prove (1) that their radical axis will intersect ON produced in the middle point R of TT'; (2) that RT^2 is equal to the product of the segments of any chord drawn from R to either circle.

Ex. 2. Show that the greatest range $r=PQ$ on any straight line PQ making a given angle θ with $OP=r_1$ is determined by $(k^2-r_1^2)/r=k-r_1\cos\theta$.

Show also that in this case $OT=k$, and $NT=NP=NQ$. Thence deduce that the common tangent at Q to the trajectory and the envelope intersects the direction of projection from P at right angles in a point T which lies on the circle whose centre is O and radius k.

The first part follows from the focal polar equation of the ellipse and the second from known geometrical properties of the ellipse.

331. Examples. *Ex.* 1. If the sun were broken up into an indefinite number of fragments, uniformly filling the sphere of which the earth's orbit is a great circle, prove that each would revolve in a year. [Coll. Ex.]

The attractions of a homogeneous solid sphere on the particles composing it are proportional to their distances from the centre.

Ex. 2. A particle moves in a conic so that the resolved part of the velocity perpendicular to the focal distance is constant, prove that the force tends to the centre of the conic. [Math. Tripos.]

Ex. 3. A particle describes an ellipse, the force tending to the centre; prove that if the circle of curvature at any point P cut the ellipse in Q, the times of transit from Q to P through A and P to Q through B are in the same ratio as the times of transit from A to P and P to B, where A and B are the extremities of the major and minor axes and P lies between A and B.

Ex. 4. A particle is attracted to a fixed point with a force μ times its distance from the point and moves in a medium in which the resistance is k times the velocity; prove that, if the particle is projected with velocity v at a distance a from the fixed point, the equation of the path when referred to axes along the initial radius and parallel to the direction of projection is

$$k\tan^{-1}2any/(2vx-aky)+n\log(x^2/a^2+\mu y^2/v^2-kxy/av)=0,$$

where $n^2=\mu-k^2/4$. [Coll. Ex. 1887.]

Ex. 5. Three centres of force of equal intensity are situated one at each corner of a triangle ABC and attract according to the direct distance. A particle moving under their combined influence describes an ellipse which touches the sides of the triangle ABC. Prove that the points of contact are the middle points of the sides, and that the velocities at these points are proportional to the sides.

[Math. Tripos, 1893.]

Ex. 6. If any number of particles be moving in an ellipse about a force in the centre, and the force suddenly cease to act, show that after the lapse of $(1/2\pi)$th part of the period of a complete revolution all the particles will be in a similar concentric and similarly situated ellipse. [Math. Tripos, 1850.]

Ex. 7. A particle moves in an ellipse under a centre of force in the centre. When the particle arrives at the extremity of the major axis the force ceases to act until the particle has moved through a distance equal to the semi-minor axis; it then acts for a quarter of the periodic time in the ellipse. Prove that if it again ceases to act for the same time as before, the particle will have arrived at the other end of the major axis. [Art. 325.] [Math. Tripos, 1860.]

Ex. 8. An elastic string passes through a smooth straight tube whose length is the natural length of the string. It is then pulled out equally at both ends until its length is increased by $\sqrt{2}$ times its original length. Two equal perfectly elastic balls are attached to the extremities and projected with equal velocities at right angles to the string, and so as to impinge on each other. Prove that the time of impact is independent of the velocity of projection, and that after impact each ball will move in a straight line, assuming that the tension of the string is proportional to the extension throughout the motion. [Math. Tripos, 1860.]

Ex. 9. A point is moving in an equiangular spiral, its acceleration always tending to the pole S; when it arrives at a point P the law of acceleration is changed to that of the direct distance, the actual acceleration being unaltered. Prove that the point P will now move in an ellipse whose axes make equal angles with SP and the tangent to the spiral at P, and that the ratio of these axes is $\tan \frac{1}{2}\alpha : 1$ where α is the angle of the spiral.

Ex. 10. A series of particles which attract one another with forces varying directly as the masses and distance are under the attraction of a fixed centre of force also varying directly as the distance; prove that if they are projected in parallel directions from points lying on a radius vector passing through the centre of force with velocities inversely proportional to their distances from the centre of force, they will at any subsequent time lie on a hyperbola. [Math. Tripos, 1888.]

Ex. 11. A particle starting from rest at a point A moves under the action of a centre of force situated at S whose magnitude is equal to μ . (distance from S). It arrives at A after an interval T and the centre of force is then suddenly transferred to some other point S' without altering its magnitude. If the particle be at a point B at the termination of a second interval T equal to the former, prove that the straight lines SS' and AB bisect each other. If at this instant the centre of force be suddenly transferred back to its original position S, prove that at the end of a third interval T the particle will be at S'. If at that instant the centre of force ceased to act, the particle will describe a path which passes through its original position A.

Ex. 12. If the central force is attractive and proportional to $u^2/(cu+\cos\theta)^3$, prove that the orbit is one of the conics given by the equation

$$(cu+\cos\theta)^2 = a + b\cos 2(\theta+\alpha). \qquad \text{[Coll. Ex. 1896.]}$$

Putting $cu+\cos\theta = U$, the differential equation of the path becomes the same as that for a central force varying as the distance $1/U$. The solution is therefore known to be the form given above.

Ex. 13. A particle moves under a central force $F = \mu u^2(1+k^2\sin^2\theta)^{-\frac{3}{2}}$. Find the orbit and interpret the result geometrically. [Math. Tripos.]

Ex. 14. A smooth horizontal plane revolves with angular velocity ω about a vertical axis to a point of which is attached the end of a weightless string, extensible according to Hooke's law and of natural length d just sufficient to reach the plane. The string is stretched and after passing through a small ring at the point where the axis meets the plane is attached to a particle of mass m which moves on the plane. Show that, if the mass be initially at rest relative to the plane, it will describe on the plane a hypocycloid generated by the rolling of a circle of radius $\frac{1}{2}a\{1-\omega(md\lambda^{-1})^{\frac{1}{2}}\}$ on a circle of radius a, where a is the initial extension and λ the coefficient of elasticity of the string.

[Math. Tripos, 1887.]

The accelerating tension is $\lambda r/md = \mu r$ (say). The path in space is therefore an ellipse having a and $b = \omega a/\sqrt{\mu}$ for semi-axes. To find the path relative to the rotating plane we apply to the particle a velocity ωr transverse to r backwards. If p' be the perpendicular from the centre on the resultant of v and ωr, we have by taking moments about the centre

$$(v^2 - 2v\omega p + \omega^2 r^2)\, p'^2 = (vp - \omega r^2)^2.$$

Substituting for v^2 and vp their values in elliptic motion we find

$$b^2 (a^2 - r^2) = p'^2 (a^2 - b^2).$$

This is a linear relation between r^2 and p'^2 and the curve will be an epicycloid if the radii of the corresponding circles are real (Art. 322). To find the radius of the fixed circle, we put $p' = 0$; this gives the radius $r = a$. To find the radius c of the rolling circle, we put $p' = r$, and $r = a + 2c$; this gives the required value of c. If c is negative the curve is a hypocycloid.

Law of the inverse square of the distance.

332. *A particle is acted on by a centre of force situated in the origin whose acceleration is $F = \mu u^2$ where u is the reciprocal of the radius vector. It is required to find the possible orbits.*

We have the differential equation (Art. 309)

$$\frac{d^2 u}{d\theta^2} + u = \frac{F}{h^2 u^2} = \frac{\mu}{h^2} \quad \text{.....................(1)};$$

$$\therefore\ u = \frac{\mu}{h^2} + A \cos(\theta - \alpha),$$

where A and α are the constants of integration. Comparing this with the equation of a conic

$$lu = 1 + e\cos(\theta - \alpha) \quad \text{.....................(2)},$$

where l is the semi-latus ectum, we see that *the orbit is a conic having one focus at the centre of force. We also have $h^2 = \mu l$.*

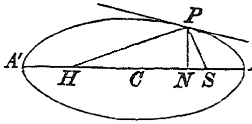
Conversely, if the orbit is a conic with the centre of force in one focus, the law of force must be the inverse square. To prove this, we let (2) be the given equation of the orbit; substituting in the left-hand side of equation (1) we find $F = \mu u^2$, where μ has been written for the constant h^2/l.

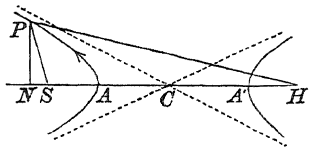
333. The velocity. The relations between the conic and the force are more easily deduced from the equation

$$F = -\tfrac{1}{2} h^2 \frac{d}{dr}\frac{1}{p^2} = \frac{\mu}{r^2},$$

the force being attractive in the standard case,

$$\therefore \frac{h^2}{p^2} = \frac{2\mu}{r} + C,$$





where C is the constant of integration. The p and r equation of an ellipse having a focus S at the origin is

$$\frac{l}{p^2} = \frac{2}{r} - \frac{1}{a},$$

where $l = b^2/a$ is the semi-latus rectum. Comparing these equations, we have the standard formulæ

$$h^2 = \mu l, \quad C = -\frac{\mu}{a}, \quad \therefore v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \text{.........(A).}$$

We change from the ellipse to the hyperbola by making the centre C pass through infinity to the other side of the origin S, we therefore put $-a'$ for a; also b^2 becomes $-b'^2$, the semi-latus rectum remaining positive and equal to b'^2/a'. We now have

$$h^2 = \mu l, \quad C = \frac{\mu}{a'}, \quad \therefore v^2 = \mu\left(\frac{2}{r} + \frac{1}{a'}\right) \text{..............(B).}$$

In passing from that branch of the hyperbola which is concave to the centre of force to the convex branch, the radius vector r changes sign through infinity from positive to negative. Before comparing the equation of the orbit with that of the hyperbola we should write $-r'$ for r in the latter. Also since this branch is convex to the origin the force is repulsive and μ is negative, let us put $\mu = -\mu'$. Comparing the formulæ

$$v^2 = \frac{h^2}{p^2} = -\frac{2\mu'}{r'} + C, \quad \frac{l}{p^2} = -\frac{2}{r'} + \frac{1}{a'},$$

we have

$$h^2 = \mu' l, \quad C = \frac{\mu'}{a'}, \quad \therefore v^2 = \mu'\left(-\frac{2}{r'} + \frac{1}{a'}\right) \text{...........(C).}$$

In the parabola, a is infinite, and

$$h^2 = \mu l, \quad C = 0, \quad v^2 = \mu\frac{2}{r} \text{................(D).}$$

All these formulæ may be included in the standard form (A) of the ellipse if we understand that on the concave branch of the hyperbola the major axis is by interpretation negative; on the convex branch, the radius vector being made positive, the major axis is positive while the semi-latus rectum l and the strength μ are negative.

334. Construction of the orbit. When the velocity V and the distance R are known at any point P of the orbit (say, the initial position), we may determine the curve in the following manner. *Let the force be attractive.* The orbit is now concave to the centre of force and μ is positive. Comparing the formulæ (A), (B) and (D) and remembering that the velocity V_1 from infinity to the initial position is given by $V_1^2 = 2\mu/R_1$ (Art. 312), we see that *the orbit is an ellipse, parabola or the concave branch of a hyperbola according as the velocity is less than, equal to, or greater than that from infinity.* We notice that this criterion is independent of the angle of projection at P. *Let the force be repulsive.* Since the path is convex to the centre of force *the orbit is the convex branch of a hyperbola.*

335. Having ascertained the nature of the orbit we have next to determine the lengths of the major axis and latus rectum. Supposing the ellipse to be the standard case, we have by (A), $\dfrac{1}{a} = \dfrac{2}{R} - \dfrac{V^2}{\mu}$. We notice that the length a is independent of the angle of projection. *If then particles are projected from the same point with equal velocities the major axes of the orbits described are equal.*

If β be the angle of projection (Art. 313) we have $p = R \sin\beta$ and $h = Vp$. The constant h and the semi-latus rectum l are therefore found from $\quad h = VR\sin\beta, \quad h^2 = \mu l.$

336. *The position in space of the major axis may be found in various ways.* Let S be the focus occupied by the centre of force and A the extremity of the major axis nearest to S.

We may find θ from the analytical equation of the curve

$$l/r = 1 + e\cos\theta,$$

where θ is the angle the initial radius vector SP makes with SA.

We may also use a geometrical construction. The focus S and the tangent PT at P being known, we can draw a straight line PH so that SP, PH make equal angles with PT, the direction of PH depending on whether the curve is an ellipse or hyperbola. If the point H is then determined so that $SP+PH=2a$, where a has been already found, it is clear that H is the empty focus. If the curve is a hyperbola, these lengths (as already explained) must have their proper signs. The position of the major axis is then found by joining S and H, and a being known the eccentricity e is equal to $SH/2a$.

337. *Ex.* 1. The initial distance of a particle from the centre of force being r, and the initial radial and transverse velocities being V_1 and V_2, prove that the latus rectum $2l$ and the angle θ which the radius vector r makes with the major axis are given by $\dfrac{l}{r^2}=\dfrac{V_2^2}{\mu}$, $\tan\theta=\dfrac{V_1V_2}{V_2^2-\mu/r}$.

Ex. 2. Prove that there are two directions in which a particle can be projected from a given point P with a given velocity V, so that the line of apses may have a given direction Sx in space, and find a geometrical construction for these directions.

Since V is given, a is known. With centre P and radius $2a-r$ describe a circle cutting Sx in H, H'. The required directions bisect externally the angles SPH, SPH'.

Let β be either of the angles the direction of projection at P makes with SP, Art. 313. The quadratic giving the two values of $\tan\beta$ is

$$\cot^2\beta+\left(2-\frac{r}{a}\right)\cot\theta\cot\beta+\frac{r}{a}-1=0,$$

where θ is the angle PSx. This follows from Ex. 1 by writing $V_1=V\cos\beta$, $V_2=V\sin\beta$. The quadratic may also be written in the form

$$\tan(\theta+\beta)=\left(\frac{r}{a}-1\right)\tan\beta.$$

Ex. 3. Three focal radii SP, SQ, SR of an elliptic orbit and the angles between them are given. Show that the ellipticity may be found from the equation $b\Delta=a\Delta'$, where Δ is the area PQR, Δ' the area of a triangle whose sides are $2SQ^{\frac{1}{2}}\,.\,SR^{\frac{1}{2}}\sin\frac{1}{2}QSR$ and two similar expressions. [Math. Tripos, 1893.]

Let P', Q', R' be the points on the auxiliary circle which correspond to P, Q, R. We first find by elementary conics the length of the side $Q'R'$ in terms of SQ, SR and the contained angle. The result shows that the side $Q'R'$ is equal to the corresponding side of the triangle Δ' after multiplication by a/b. Since the areas of the triangles PQR, $P'Q'R'$ are known to be in the ratio b/a, the result follows at once.

Ex. 4. Two particles P, Q describe the same orbit about a centre of force O. Prove that throughout the motion the area contained by the radii vectores OP, OQ is constant.

Thence deduce that if a ring of meteors (not attracting each other) describe a closed orbit, the angular distance between consecutive meteors varies inversely as the square of their distance from O.

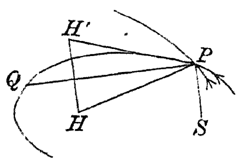
Ex. 5. Two particles P, Q describe adjacent elliptic orbits of small eccentricity in equal times, the centre of force being in the focus and the major axes coincident in direction. Supposing the particles to be simultaneously at corresponding apses, prove that the angle ψ which PQ makes with the line of apses is given by $\cot\psi = -3\operatorname{cosec} 2nt + \cot 2nt$, and find when ψ is a maximum.

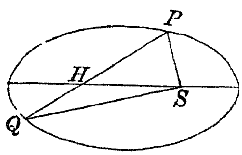
338. Elements of an orbit. To fix the position in space of an elliptic orbit described about a focus we must know the values of *six constants*, called the elements of the orbit.

These are (1) the angle which the radius vector from the given focus to the nearer extremity of the major axis makes with some determinate line in the plane of the orbit, the angle being measured in the positive direction; (2) the length of the major axis; (3) the eccentricity; (4) a constant usually called the epoch to fix the longitude of the particle at the time $t = 0$. This constant will be considered later on.

To determine the plane of the orbit we require two more constants. Taking the focus as origin, let some rectangular axes be given in position. Let the plane of the orbit intersect the plane of xy in the straight line $N'SN$. This line is called the line of nodes, and that node at which the particle passes to the positive side of the plane of xy is called the ascending node. We require (5) the angle the radius vector to the ascending node makes with the axis of x, and (6) the inclination of the plane of the orbit to the plane of xy.

339. Point to Point. *To project a particle with a given velocity V from a given point P so that it shall pass through another given point Q.*





Let r_1, r_2 be the distances SP, SQ. The velocity at P being given, the major axis $2a$ is also known from the formula $V^2 = \mu\left(\frac{2}{r_1} - \frac{1}{a}\right)$.

With centres P and Q, describe two circles of radii $2a - r_1$, $2a - r_2$; these intersect in two points H, H'. Either of these may be the empty focus. *The three sides of the equal triangles PQH, PQH' are therefore known.*

There are two directions of projection which satisfy the given conditions. These directions are the bisectors of the supplements of the angles SPH, SPH'. Let β, β' be the angles of projection at P (measured behind the radius vector SP, see Art. 313), then $\beta + \beta'$ is equal to the supplement of SPQ, and $\beta - \beta'$ is equal to the known angle HPQ.

The range PQ on a given straight line is the greatest possible when H, H' coincide and lie on the straight line PQ. We then have

$$PQ = PH + QH = 4a - r_1 - r_2.$$

This equation requires that the semi-major axis should be one-quarter of the perimeter of the triangle SPQ.

Since two consecutive trajectories whose foci are in the neighbourhood of PQ intersect in Q, the locus of Q as the range PQ turns round P is *the envelope of all trajectories from a given point P with a given velocity. Since $PQ + QS = 4a - r_1$ this locus is another ellipse having its foci at P and S.* Each trajectory touches the enveloping ellipse in the point where the straight line joining P to the empty focus of the trajectory cuts either curve.

340. *Ex.* 1. Prove that the semi-major axis a', the eccentricity e' and the semi-latus rectum l' of the enveloping ellipse are given by

$$2a' = 4a - r_1, \qquad e' = \frac{r_1}{4a - r_1}, \qquad l'^2 = 2a\,(2a - r_1).$$

Ex. 2. If the variation of gravity is taken account of and the resistance of the air neglected, prove that the least velocity with which a shot could be projected from the pole so as to meet the earth's surface at the equator is about $4\frac{1}{2}$ miles per second, and that the angle of elevation is $22\frac{1}{2}°$. [Coll. Ex. 1892.]

Ex. 3. If a particle when projected from P_1 passes through two other points P_2, P_3, prove that the semi-latus rectum l is given by either of the equalities

$$l\Delta = r_1A_1 + r_2A_2 + r_3A_3 = 2r_1r_2r_3 \sin\alpha_1 \sin\alpha_2 \sin\alpha_3,$$

where r_1, r_2, r_3, are the distances SP_1, SP_2, SP_3; A_1, A_2, A_3 are the areas of the triangles P_2SP_3, P_3SP_1, P_1SP_2; α_1, α_2, α_3 the angles at the focus S and Δ is the area of the triangle $P_1P_2P_3$. Prove also that the eccentricity is given by

$$e^2\,(\Sigma A)^2 = \Sigma\,(A\,.\,\sec\alpha)^2 - 2\Sigma\,(A_1A_2 \sec\alpha_1 \sec\alpha_2 \cos\alpha_3).$$

341. Time of describing any arc. The time of describing the whole ellipse, usually called the periodic time, can be deduced

at once from the formula $A = \frac{1}{2}ht$, (Art. 306). Putting $A = \pi ab$ and $h^2 = \mu b^2/a$, (Art. 332), we find that *the periodic time* $= \dfrac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}}$.

It appears from this that the period is independent of the minor axis and depends only on the strength μ of the centre of force and on the length of the major axis.

If n be the mean angular velocity in the orbit, the mean being taken with regard to time, the period is $2\pi/n$. It follows that $n^2 = \dfrac{\mu}{a^3}$.

342. *To find the time of describing any arc AP of an elliptic orbit.*

Let S be the focus occupied by the centre of force, AQA' the auxiliary circle and QPN an ordinate. If A is the extremity of the major axis nearest to S, the angle ASP is called *the true anomaly* and is sometimes represented by the letter v, i.e. the angle $ASP = v$. The angle ACQ is the eccentric angle of P and in astronomy is called *the eccentric anomaly*; it is usually represented by u, i.e. the angle $ACQ = u$. Thus the true anomaly v is measured at the centre of force, the eccentric anomaly u at the centre of the orbit.

When the particle is a planet the extremities A, A' of the major axis are called *the perihelion* and *aphelion*; when the particle is the moon the same points are called perigee and apogee. They are also called *the apses*, Art. 314.

Representing the time of describing the arc AP by t, and the mean angular velocity of the particle by n, the product nt is called *the mean anomaly*, and is generally represented by m, i.e. $m = nt$. To represent this angle geometrically we let a second particle describe a circle, having its centre at S, with a uniform motion in the same period as the given particle describes the ellipse. The actual angular velocity of this particle is therefore n. If A and Q' are its positions at the times $t = 0$ and $t = t$, the angle $ASQ' = nt$.

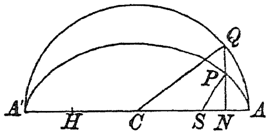
The true and mean anomalies are the important angles in the theory of elliptic motion. The eccentric anomaly is introduced as an auxiliary angle because, by its help, very simple expressions can be found for the other two anomalies and for the radius vector.

The difference between the true and the mean anomaly, or $v - m$, is called *the equation of the centre*, and is positive from the nearer apse to the farther and negative from the farther to the nearer.

Using the geometrical theorem that the ratio of the area ASP of the ellipse to the corresponding area ASQ of the circle is constant for all positions of P and equal to b/a, we have, if $A = \text{area } ASP$,

$$A = \frac{b}{a}(\text{area } ACQ - \text{area } SCQ)$$

$$= \frac{1}{2}\frac{b}{a}(a^2 u - a^2 e \sin u).$$



Since $A = \frac{1}{2}ht$, $h^2 = \mu b^2/a$, $n^2 = \mu/a^3$, this gives

$$nt = u - e\sin u \qquad \text{.....(A)}.$$

We may obtain this relation between u and t without using any figure. Taking the focus S for origin, we have

$$x' = -ae + a\cos u, \qquad y' = b\sin u,$$
$$hdt = 2dA = x'dy' - y'dx'.$$

Substituting for x' and y' we obtain t in terms of u by an easy integration.

343. To find the relation between the true and eccentric anomalies we notice that $CS = ae$, $CN = x$, $SP = r = a - ex$.

$$\therefore\ 1 - \cos v = 1 + \frac{ae - x}{a - ex} = \frac{(1+e)(a-x)}{r},$$

$$1 + \cos v = 1 - \frac{ae - x}{a - ex} = \frac{(1-e)(a+x)}{r}.$$

Remembering that $x = a\cos u$, these give at once

$$\sqrt{\frac{r}{a}}\sin\frac{v}{2} = \sqrt{(1+e)}\sin\frac{u}{2}, \qquad \sqrt{\frac{r}{a}}\cos\frac{v}{2} = \sqrt{(1-e)}\cos\frac{u}{2};$$

$$\therefore\ \tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2} \qquad \text{.................(B)}.$$

Eliminating u between (A) and (B) we have

$$nt = 2\tan^{-1}\left\{\sqrt{\frac{1-e}{1+e}}\tan\frac{v}{2}\right\} - e\sqrt{(1-e^2)}\frac{\sin v}{1 + e\cos v}.$$

The expression for the time in terms of the longitude θ may also be found by integration. Since $r^2 d\theta/dt = h$, we have $t = \frac{l^2}{h}\int \frac{f^2 d\theta}{(f + \cos\theta)^2}$, where $f = 1/e$. But it

is known that $\int \frac{d\theta}{f+\cos\theta} = \frac{2}{\sqrt{(f^2-1)}} \tan^{-1}\left(\sqrt{\frac{f-1}{f+1}} \tan\frac{\theta}{2}\right)$. By differentiating this with regard to f, the value of t follows at once.

Ex. Prove that $r\frac{dv}{du} = \sqrt{(al)}$, and $r\frac{du}{dt} = an$.

344. *Ex.* 1. Prove that the mean distance of a planet from the sun is a or $a(1+\frac{1}{2}e^2)$ according as the mean is taken with reference to the longitude or the time. [These means are respectively $\int r d\theta/2\pi$ and $\int r dt/T$, where T is the periodic time.]

Ex. 2. Prove that the mean value of r^n with regard to time for a planet is $\frac{a^n}{L(n+1)} \frac{(f^2-1)^{n+3/2}}{(-f)^{n+1}} \frac{d^{n+1}}{df^{n+1}} (f^2-1)^{-1/2}$, where $f=1/e$ and $L(n)=1 \cdot 2 \cdot 3 \ldots n$.

Ex. 3. The earth's orbit being regarded as a circle, prove that a comet, describing a parabolic orbit in the same plane, cannot remain within the circumference of the earth's orbit longer than the $(2/3\pi)$th part of a year. [Coll. Ex.]

Ex. 4. A particle is projected from the earth's surface so as to describe a portion of an ellipse whose major axis is $1\frac{1}{2}$ times the earth's radius. If the direction of projection make an angle of $30°$ with the vertical, prove that the time of flight is $\frac{3}{4}(3a/g)^{\frac{1}{2}}\{\tan^{-1}\sqrt{6}+\sqrt{\frac{2}{3}}\}$ where a is the earth's radius.

[Coll. Ex. 1895.]

345. Orbits of small eccentricity. The equations (A) and (B) of Arts. 342, 343 determine the time of describing any given angle v in an elliptic orbit of any eccentricity, the equation (B) giving u when v is known while the equation (A) then determines t. The converse problem of finding the polar coordinates r and v when t is given is usually called *Kepler's problem.* One solution by which u and v are expressed in terms of t by series arranged in ascending powers of e will be presently considered. It is enough here to notice that in a planetary orbit, where e is small, the value of u when t is given can be found by successive approximation. The value of v then follows from (B) by using the trigonometrical tables.

346. To solve $\phi(u) = u - e\sin u - m = 0$ by Newton's rule, when m, i.e. nt, is given.

Supposing u_1, u_2 to be two successive approximations to the value of u, that rule gives

$$u_2 - u_1 = -\frac{\phi(u_1)}{\phi'(u_1)} = \frac{m-m_1}{1-e\cos u_1},$$

where $m_1 = u_1 - e\sin u_1$. To find a first approximation we notice that u lies between m and $m \pm e$, the upper or lower sign being taken according as m is $<\pi$ or $>$. We choose some value of u, lying between these limits, which is an integer number of minutes so that its trigonometrical functions can be found from the tables without interpolation. By Fourier's addition to Newton's rule this first approximation should be such that $\phi(u)$ and $\phi''(u)$ have the same sign.

Substituting this first approximation for u_1, the formula gives a second approximation. Substituting again this second approximation for u_1, we obtain a third, and so on. When e is very small the first computed value of the denominator is sometimes sufficiently accurate for all the approximations required. See Encke, *Berliner Astronomisches Jahrbuch*, 1838. Gauss, *Theoria Motus* &c., translated by C. H. Davis. Adams's *Collected Works*, vol. I. p. 289.

Ex. Prove that if we choose $u_1 = m + e$ as the first approximation, the error of the value of u_2 is of the order e^3.

347. *Ex.* 1. *Leverrier's rule.* If terms of the order e^4 can be neglected, prove

$$u = m + \frac{e \sin m}{1 - e \cos m} - \frac{1}{2}\left(\frac{e \sin m}{1 - e \cos m}\right)^3.$$

Glaisher remarks that if we replace the third term by $-\frac{1}{2}(e \sin m)^3 (1 - e \cos m)^{-\frac{10}{3}}$ the formula is correct when terms of the order e^5 are neglected. He also gives a series for u correct up to e^8. *Monthly Notices of the Astronomical Society*, 1877.

Ex. 2. Prove that $\cot u = \cot m - \dfrac{e \operatorname{cosec} m}{f(u - m)}$ where

$$f(x) = \frac{x}{\sin x} = 1 + \frac{1}{6}\sin^2 x + \frac{3}{40}\sin^4 x + \frac{15}{336}\sin^6 x + \&c.$$

Putting $u = m + e$ on the right-hand side of the first equation we obtain an approximation for $\cot u$ whose error is of the order e^3. This is Zenger's solution of Kepler's problem. He has tabulated the values of $f(e)$ for the eight principal planets. Some improvements of the method have been suggested by J. C. Adams. Both papers are to be found in the *Monthly Notices of the Astronomical Society*, 1882, vol. XLII. p. 446, vol. XLIII. p. 47.

Ex. 3. Prove the following graphical solution of Kepler's problem. Construct the curve of sines $y = \sin x$, measure a distance $OM = m$ along the axis of x and draw MP making the angle PMx equal to $\cot^{-1} e$. If MP cut the curve in P, the abscissa of P is the value of u.

This method was described by J. C. Adams at the meeting of the B. Association in 1849. It is also given by See in the *Astronomical Notices*, 1895, who also refers to Klinkerfues and Dubois. Another graphical solution, using a trochoid, is given by Plummer, *Astronomical Notices*, 1895, 1896.

Ex. 4. The equation $u - e \sin u = m$ has only one real value of u when m is given.

This follows from the graphical construction. If the ordinate MP could cut the curve in a second point Q, move the straight line PQ parallel to itself until P and Q coincide. We should then have a tangent to the curve making an angle $\tan^{-1} 1/e$ with the axis of x. But if $e < 1$ this is impossible, for in the curve of sines the greatest value of the angle is 45°.

Ex. 5. By using Lagrange's theorem we may expand $f(u)$ in a series of ascending powers of the eccentricity, the coefficients being functions of m. Prove that if the form of the function $f(u)$ be so chosen that the coefficient of e^2 is zero, we obtain the series

$$\cot u = \cot m - e \operatorname{cosec} m + \tfrac{1}{6} e^3 \sin m + \&c.,$$

which takes a very simple form, when the cubes of e can be neglected. This equation is due to Rob. Bryant, *Astronomical Notices*, 1886.

Ex. 6. Prove that when e^4 can be neglected

$$\sin \tfrac{1}{2}(u - m) = \tfrac{1}{2} e \sin m + \tfrac{1}{4} e^2 \sin 2m + \tfrac{5}{12} e^3 \sin 3m + \&c. \qquad \text{[R. Bryant.]}$$

Ex. 7. If θ' be the longitude of a planet seen from the empty focus and measured from an apse, prove that

$$\theta' = nt + \tfrac{1}{4} e^2 \sin 2nt + \&c.,$$

the error being of the order e^4. It follows that the angular velocity round the empty focus is very nearly constant.

348. We may apply the method of Art. 342 to find the time of describing an arc of the concave branch of the hyperbola. Taking the focus as origin the equation of a hyperbola may be written

$$x'=ae-\frac{a}{2}(f^u+f^{-u}),\qquad y'=\frac{b}{2}(f^u-f^{-u}),$$

where u is an auxiliary quantity and f a constant which will be immediately chosen to be the base of the Napierian logarithms;

$$\therefore\ h\,dt=2dA=x'dy'-y'dx'=ab\left\{\frac{e}{2}(f^u+f^{-u})-1\right\}du.$$

Since $h^2=\mu b^2/a$ we have, putting $\mu/a^3=n^2$,

$$nt=-u+e\sinh u \quad\text{.................................(A)}.$$

Again, as in Art. 343, we have $x=CN=\frac{a}{2}(f^u+f^{-u})$;

$$\therefore\ \cos v=\frac{ae-x}{ex-a};\quad \therefore\ \tan\frac{v}{2}=\sqrt{\frac{e+1}{e-1}}\tanh\frac{u}{2} \quad\text{.............. (B)},$$

where $v=\angle ASP$. If we eliminate u, we have

$$nt=\log\frac{\sqrt{(e+1)}-\sqrt{(e-1)}\tan\frac{1}{2}v}{\sqrt{(e+1)}+\sqrt{(e-1)}\tan\frac{1}{2}v}+e\sqrt{(e^2-1)}\frac{\sin v}{1+e\cos v}.$$

To find a geometrical interpretation for the auxiliary quantity u, let us describe a rectangular hyperbola having the same major axis and produce the ordinate NP to cut the rectangular hyperbola in Q. Then $\tan QCN=\tanh u$.

Ex. A particle describes the convex branch of the hyperbola, and $\mu=-\mu'$ is negative. Prove

$$nt=u+e\sinh u,\qquad \tan\frac{v}{2}=\sqrt{\frac{e-1}{e+1}}\tanh\frac{u}{2},$$

where $v=ASP$, $\mu'/a^3=n^2$

349. The time in a parabolic orbit may be more easily found by using the equation $r^2d\theta=hdt$.

Putting $l/r=1+\cos v$ where l is the semi-latus rectum, and $h^2=\mu l$, we have

$$\sqrt{\frac{\mu}{l^3}}\,t=\int\frac{dv}{(1+\cos v)^2}=\frac{1}{2}\int\left(1+\tan^2\frac{v}{2}\right)d\tan\frac{v}{2}$$
$$=\frac{1}{2}\left(\tan\frac{v}{2}+\frac{1}{3}\tan^3\frac{v}{2}\right).$$

This formula gives the time t of describing the true anomaly $v=ASP$.

If c be the radius of the earth's orbit, and p the perihelion distance of the particle *expressed as a fraction of* c, we have $l=2pc$. To eliminate μ, let $T=2\pi\sqrt{c^3/\mu}$ be the length of a year. Then

$$\frac{\pi\sqrt{2}}{T}\,.\,t=p^{\frac{3}{2}}\left\{\tan\frac{v}{2}+\frac{1}{3}\tan^3\frac{v}{2}\right\}.$$

If we write $T=365{\cdot}256$ this gives t in days.

When a formula like this has to be frequently used we construct a table to save the continual repetition of the same arithmetical work. Let the values of $\{\tan\frac{1}{2}v + \frac{1}{3}\tan^3\frac{1}{2}v\}$ be calculated for values of v from 0 to 180°, with differences for interpolation. When p is known for any comet moving in a parabolic orbit, the table can be used with equal ease to find the time when the true anomaly is given or the true anomaly when the time is known.

350. Euler's theorem. *A particle describes a parabola under the action of a centre of force in the focus S. It is required to prove that the time of describing an arc PP' is given by*

$$6\sqrt{\mu}t = (r + r' + k)^{\frac{3}{2}} - (r + r' - k)^{\frac{3}{2}},$$

where r, r' are the focal distances of P, P' and k is the chord joining P, P'.

Let x, y; x', y' be the coordinates of P, P', then since $y^2 = 4ax$,

$$k^2 = (x - x')^2 + (y - y')^2 = (y - y')^2 \left\{1 + \left(\frac{y + y'}{4a}\right)^2\right\}.$$

As we wish to make the right-hand side a perfect square, we put

$$y + y' = 4a\tan\theta, \quad y - y' = 4a\tan\phi \;\ldots\ldots\ldots\ldots(1).$$

We shall suppose that in the standard case y is positive and y' numerically less than y; then θ and ϕ are positive,

$$\therefore\; k = 4a\tan\phi\sec\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Also $$r + r' = 2a + x + x' = 2a(\sec^2\theta + \tan^2\phi);$$

$$\left.\begin{aligned} \therefore\; r + r' + k &= 2a(\sec\theta + \tan\phi)^2 \\ r + r' - k &= 2a(\sec\theta - \tan\phi)^2 \end{aligned}\right\};$$

$$\begin{aligned} \therefore\; (r + r' + k)^{\frac{3}{2}} &- (r + r' - k)^{\frac{3}{2}} \\ &= (2a)^{\frac{3}{2}}\{(\sec\theta + \tan\phi)^3 - (\sec\theta - \tan\phi)^3\} \\ &= 2(2a)^{\frac{3}{2}}\{3 + 3\tan^2\theta + \tan^2\phi\}\tan\phi. \end{aligned}$$

Drawing the ordinates PN, $P'N'$, we see that

$$\begin{aligned} \text{area } PSP' &= APN - AP'N' + SP'N' - SPN \\ &= \tfrac{2}{3}(xy - x'y') + \tfrac{1}{2}(x' - a)y' - \tfrac{1}{2}(x - a)y \\ &= \frac{1}{24a}(y^3 - y'^3) + \frac{a}{2}(y - y') \\ &= \tfrac{2}{3}a^2\tan\phi\{3\tan^2\theta + \tan^2\phi + 3\}. \end{aligned}$$

Since the area $PSP' = \frac{1}{2}ht = \frac{1}{2}\sqrt{(2a\mu)}\,t$ the result to be proved follows at once.

The arc PP' gradually increases as P' moves towards and past the apse. The quantity $r+r'-k$ decreases and vanishes when the chord passes through the focus. To determine whether the radical changes sign we notice that this can happen only when it vanishes. We can therefore without loss of generality so move the points P, P', that, when the chord crosses the focus, PP' is a double ordinate. We then have

$$6\sqrt{\mu}t = (2r+2y)^{\frac{3}{2}} - (2r-2y)^{\frac{3}{2}} = \{(2a+y)^3 \pm (2a-y)^3\}/(2a)^{\frac{3}{2}}.$$

Comparing this with the ordinary parabolic expression for twice the area ASP it is evident that the last term should change sign where y increases past $2a$ and that the double sign should be a minus. *The second radical in Euler's equation must be taken positively when the angle PSP' is greater than* 180°.

351. *Ex.* 1. If the ordinate $P'N'$ cut the parabola again in Q'; prove that θ, ϕ are the acute angles made by the chords PP', PQ' with the axis of y.

Ex. 2. Show that there are two parabolas which can pass through the given points P, P', and have the same focus. Show also that in using Euler's theorem to find the time P to P', the second radical has opposite signs in the two paths.

To find the parabolas we describe two circles, centres P, P' and radii SP, SP'. These circles intersect in S and the two real common tangents are the directrices. These tangents intersect on PP' and make equal angles with it on opposite sides. The concavities of the parabolas are in opposite directions, and the angles described are PSP' and $360° - PSP'$. If then one angle is greater than 180°, the other must be less.

Ex. 3. A parabolic path is described about the focus. Show that the squares of the times of describing arcs cut off by focal chords are proportional to the cubes of the chords.

352. Lambert's Theorem*. *If t is the time of describing any arc $P'P$ of an ellipse, and k is the chord of the arc, then*

$$nt = (\phi - \sin\phi) - (\phi' - \sin\phi'),$$

where $$\sin\tfrac{1}{2}\phi = \tfrac{1}{2}\sqrt{\frac{r+r'+k}{a}}, \quad \sin\tfrac{1}{2}\phi' = \tfrac{1}{2}\sqrt{\frac{r+r'-k}{a}}, \quad \text{.........(A)}.$$

Let u, u' be the eccentric anomalies of P, P',

$$\therefore\ k^2 = a^2(\cos u - \cos u')^2 + a^2(1-e^2)(\sin u - \sin u')^2$$
$$= 4a^2\sin^2\tfrac{1}{2}(u-u')\{1 - e^2\cos^2\tfrac{1}{2}(u+u')\} \text{ .... (1)},$$

* This proof of Lambert's theorem is due to J. C. Adams, *British Association Report*, 1877, or *Collected Works*, p. 410. He also gives the corresponding theorem for the hyperbola, using hyperbolic sines. In the *Astronomical Notices*, vol. XXIX., 1869, Cayley gives a discussion of the signs of the angles ϕ, ϕ'. The theorem for the parabola was discovered by Euler (*Miscell. Berolin.* t. VII.), but the extension to the other conic sections is due to Lambert.

$$r+r'=2a-ae\cos u-ae\cos u'$$
$$=2a\{1-e\cos\tfrac{1}{2}(u+u')\cos\tfrac{1}{2}(u-u')\} \quad\text{.......................}\quad (2),$$
$$nt=u-u'-e(\sin u-\sin u')$$
$$=u-u'-2e\cos\tfrac{1}{2}(u+u')\sin\tfrac{1}{2}(u-u') \quad\text{.......................}\quad (3).$$

Hence we see that if a, and therefore also n, are given, then $r+r'$, k, and t are functions of the two quantities $u-u'$, and $e\cos\frac{1}{2}(u+u')$. Let

$$u-u'=2\alpha, \qquad e\cos\tfrac{1}{2}(u+u')=\cos\beta \quad\text{.......................}\quad (4).$$
$$\therefore\ k=2a\sin\alpha\sin\beta \quad\text{................................}\quad (5),$$
$$r+r'+k=2a\{1-\cos(\beta+\alpha)\} \quad\text{..........................}\quad (6),$$
$$r+r'-k=2a\{1-\cos(\beta-\alpha)\} \quad\text{..........................}\quad (7),$$
$$nt=2\alpha-2\sin\alpha\cos\beta \quad\text{.............................}\quad (8).$$

If we put $\beta+\alpha=\phi$, $\beta-\alpha=\phi'$, the equations (6) and (7) lead to the expressions for $\sin\frac{1}{2}\phi$, $\sin\frac{1}{2}\phi'$ given above, while (8) when put into the form

$$nt=\{\beta+\alpha-\sin(\beta+\alpha)\}-\{\beta-\alpha-\sin(\beta-\alpha)\}$$

gives at once the required value of nt.

353. Let us trace the values of ϕ, ϕ' as the point P travels round the ellipse in the positive direction beginning at a fixed point P'. We suppose that u increases from u' to $2\pi+u'$.

The positive sign has been given to the square root k. Since k can vanish only when P coincides with P', and α begins positively, we see that both α and β lie between 0 and π for all positions of P. The latter is also restricted to lie between $\cos^{-1}e$ and $\pi-\cos^{-1}e$.

We have by differentiating (4)

$$d\phi=d\beta+d\alpha=\ \ \tfrac{1}{2}du\{1+e\operatorname{cosec}\beta\sin\tfrac{1}{2}(u+u')\},$$
$$d\phi'=d\beta-d\alpha=-\tfrac{1}{2}du\{1-e\operatorname{cosec}\beta\sin\tfrac{1}{2}(u+u')\}.$$

Since $\sin^2\beta=e^2\sin^2\frac{1}{2}(u+u')+1-e^2$, and $e^2<1$, it follows that $d\phi$ is always positive and $d\phi'$ always negative. If β_0 be the least value of β which satisfies $\cos\beta=e\cos u'$, ϕ continually increases from β_0 to $2\pi-\beta_0$ and ϕ' decreases from β_0 to $-\beta_0$.

When $\phi=\pi$, $r+r'+k=4a$, and the chord $P'P$ passes through the empty focus H. Let it cut the ellipse in Q. It follows that ϕ is less or greater than π according as P lies in the arc $P'Q$ or QP'.

When $\phi'=0$, $r+r'-k=0$, and the chord $P'P$ passes through the centre of force S. Let it cut the ellipse in R. Then ϕ' is positive or negative according as P lies in the arc $P'R$ or RP'.

The values of ϕ, ϕ' are determined by the radicals (A). Each of these gives more than one value of the angle, thus ϕ may be greater or less than π and ϕ' may be positive or negative. This ambiguity disappears (as explained above) when the position of P on the ellipse is known. Thus $\sin\phi$ and $\sin\phi'$ have the same sign when the two foci are on the same side of the chord PP' and opposite signs when the chord passes between the foci.

354. *Ex.* 1. Prove that the time t of describing an arc $P'P$ of a hyperbola is given by

$$t\sqrt{\frac{\mu}{a^3}}=-\phi+\phi'+\sinh\phi-\sinh\phi',$$

where $\sinh\frac{\phi}{2}=\sqrt{\frac{r+r'+k}{4a}}$, $\sinh\frac{\phi'}{2}=\sqrt{\frac{r+r'-k}{4a}}$,

and k is the chord of the arc. [Adams.]

Ex. 2. The length of the major axis being given, two ellipses can be drawn through the given points P, P' and having one focus at the centre of force. Prove that the times of describing these arcs, as given by Lambert's theorem, are in general unequal.

To find the ellipses we describe two circles with the centres at P, P' and the radii equal to $2a-SP$, and $2a-SP'$. These intersect in two points H, H', either of which may be the empty focus, and these lie on opposite sides of the chord PP'.

355. Two centres of force. *Ex.* 1. An ellipse is described under the action of two centres of force, one in each focus. If these forces are $F_1(r_1)$ and $F_2(r_2)$, prove that $\frac{1}{r_1^2}\frac{d}{dr_1}(r_1^2F_1)=\frac{1}{r_2^2}\frac{d}{dr_2}(r_2^2F_2)$. If one force follow the Newtonian law, prove that the other must do so also.

These results follow from the normal and tangential resolutions.

Ex. 2. A particle describes an elliptic orbit under the influence of two equal forces, one directed to each focus. Show that the force varies inversely as the product of the distances of the particle from the foci. [Coll. Ex.]

Ex. 3. A particle describes an ellipse under two forces tending to the foci, which are one to another at any point inversely as the focal distances; prove that the velocity varies as the perpendicular from the centre on the tangent, and that the periodic time is $\pi(a^2+b^2)/kab$, ka, kb being the velocities at the extremities of the axes. [Coll. Ex.]

Ex. 4. A particle describes an ellipse under the simultaneous action of two centres of force situated in the two foci and each varying as (distance)$^{-2}$. Prove that the relation between the time and the eccentric anomaly is

$$\left(\frac{du}{dt}\right)^2=\frac{\mu}{a^3}\frac{1}{(1-e\cos u)^2}+\frac{\mu'}{a^3}\frac{1}{(1+e\cos u)^2}.$$

[Cayley, *Math. Messenger*, 1871.]

The inverse cube and the inverse n^{th} powers of the distance.

356. The law of the inverse cube. *A particle projected in any given manner describes an orbit about a centre of force whose attraction varies as the inverse cube of the distance. It is required to find the motion*.*

* The orbits when the force $F=\mu u^3$ were first completely discussed by Cotes in the *Harmonia Mensurarum* (1722) and the curves have consequently been called Cotes' spirals. The motion for $F=\mu u^n$ when the velocity is equal to that from infinity is generally given in treatises on this subject. The paths for several other laws of force are considered by Legendre (*Théorie des Fonctions Elliptiques*, 1825), and by Stader (*Crelle*, 1852); see also Cayley's *Report to the British Association*, 1863. Some special paths when $F=\mu u^n$, for integer values of n from $n=4$ to $n=9$, are discussed by Greenhill (*Proceedings of the Mathematical Society*, 1888), one case when $n=5$, being given in Tait and Steele's *Dynamics*.

Let attraction be taken as the standard case and let the accelerating force be $F = \mu u^3$. We have

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2} = \frac{\mu}{h^2}u,$$

$$\therefore \frac{d^2u}{d\theta^2} + \left(1 - \frac{\mu}{h^2}\right)u = 0.$$

The solution depends on the sign of the coefficient of u. Let V be the velocity of the particle at any point of its path (say the point of projection), β the angle and R the distance of projection, then $h = VR\sin\beta$; (Art. 313). Let V_1 be the velocity from infinity, then $V_1^2 = \mu/R^2$. It follows that h^2 is $>$ or $< \mu$ according as $V\sin\beta$ is $>$ or $< V_1$; i.e. the coefficient of u is positive or negative according as the transverse velocity at any point is greater or less than the velocity from infinity. If the force is repulsive the coefficient is always positive.

Case 1. Let $h^2 > \mu$, we put $1 - \mu/h^2 = n^2$, then $n < 1$ or > 1 according as the force is attractive or repulsive. The equation of the path is (Art. 119)

$$u = a\cos n(\theta - \alpha).$$

The curve consists of a series of branches tending to asymptotes, each of which makes an angle π/n with the next.

When the curve is given the motion may be deduced from the following relations (Art. 306),

$$h^2 = \frac{\mu}{1-n^2}, \qquad v^2 = \mu\left(\frac{a^2n^2}{1-n^2} + u^2\right).$$

Also by integrating $d\theta/dt = hu^2$, and putting $a = 1/b$, we find that the time of describing the angle $\theta = \alpha$ to θ, i.e. $r = b$ to r, is given by

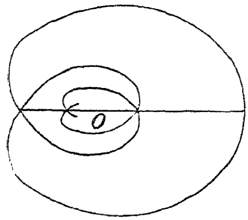
$$\tan n(\theta - \alpha) = \frac{hnt}{b^2}, \qquad r^2 - b^2 = \frac{h^2n^2t^2}{b^2}.$$

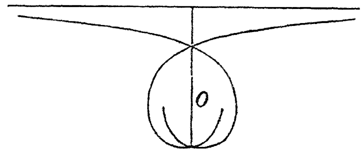
357. Case 2. Let μ be positive and $> h^2$, we put $1 - \mu/h^2 = -n^2$. The equation of the path is then $u = Ae^{n\theta} + Be^{-n\theta}$. The values of the constants A, B are to be deduced from the initial values of u and $du/d\theta$. Two cases therefore arise, according as A and B have the same or opposite signs. In the former case, u cannot vanish and therefore the orbit has no branches which go to infinity; in the latter case there is an asymptote. If we write $\theta = \theta_1 + \alpha$ and choose α so that $Ae^{n\alpha} = \mp Be^{-n\alpha}$, we may reduce

the equation to one of the three standard forms

$$u = \frac{a}{2}(e^{n\theta_1} \pm e^{-n\theta_1}), \quad u = Ae^{n\theta},$$

where $2n\alpha = \log(\pm B/A)$, $a = 2\sqrt{(\pm AB)}$, the upper or lower signs being taken according as A, B have the same or opposite signs.





The third case occurs when $B = 0$; the orbit is then the equiangular spiral already considered in Art. 319.

When the curve is given the motion may be deduced from the following relations

$$h^2 = \frac{\mu}{1+n^2}, \quad v^2 = \mu\left(\mp \frac{a^2n^2}{1+n^2} + u^2\right), \quad b^2 \mp r^2 = \left(\frac{nht}{b} + C\right)^2,$$

where C is determined by making t vanish when r has its initial value and $b = 1/a$.

When A and B have the same sign the two branches beginning at the point $\theta_1 = 0$, i.e. $\theta = \alpha$, wind symmetrically round the origin in opposite directions. When A and B have opposite signs the two branches begin at opposite ends of an asymptote, whose distance from the origin is $y = 1/an$, and then wind round the origin. As the particle approaches the centre of force, the convolutions of either branch become more and more nearly those of an equiangular spiral whose angle is given by $\cot\phi = \pm n$, the upper or lower sign being taken according as $\theta = \pm\infty$. The particle arrives at the pole with an infinite velocity at the end of a finite time.

358. Case 3. Let μ be positive and $= h^2$. The orbit is

$$u = a(\theta - \alpha).$$

When the path is known the motion is given by

$$h^2 = \mu, \quad v^2 = \mu(u^2 + a^2), \quad t\sqrt{\mu} = br,$$

where t is the time from a distance r to the centre of force and $b = 1/a$. We notice that the radial velocity is constant.

Beginning at the opposite extremities of an asymptote the two branches wind round the origin and ultimately when $\theta = \pm\infty$ cut the radius vector at right angles. If OZ is drawn perpendicular to the radius vector OP to meet the tangent at P in Z, we may show that OZ is constant and equal to $1/a$.

359. *Ex.* The motion for a force $F=f(u)$ being known, show how to deduce that for a force $F=f(u)+\mu u^3$ and give a geometrical interpretation. [Newton.]

The differential equations are

$$\frac{d^2u}{d\theta^2}+\left(1-\frac{\mu}{h^2}\right)u=\frac{f(u)}{h^2}, \qquad \frac{d\theta}{dt}=hu^2.$$

These may be reduced to the forms used when $F=f(u)$ by writing $c\theta=\theta'$, $ch=h'$, where $c^2=1-\mu/h^2$.

To construct the path $u=\phi(c\theta)$, when $u=\phi(\theta)$ is known, we make the axis of x together with the latter curve revolve round the centre of force with an angular velocity $d\omega/dt$, where $c\theta=\theta-\omega$. The axis of x therefore advances or regredes according as c is less or greater than unity.

360. Law of the inverse nth power. *It is required to find the path of a particle when the central force $F=\mu u^n$.* See Art. 320. We have

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}=\frac{\mu}{h^2}u^{n-2};$$

$$\therefore\ v^2=h^2\left\{\left(\frac{du}{d\theta}\right)^2+u^2\right\}=\frac{2\mu}{n-1}u^{n-1}+C \quad\ldots\ldots\ldots\ldots (1),$$

except when $n=1$, for then the right-hand side takes a logarithmic form.

The integration of this equation can be reduced to elementary forms when $C=0$; this requires that $n>1$ for otherwise v^2 would be negative. The equation then shows that *at every point of the orbit the velocity is equal to that from infinity*, Art. 312.

If V be the velocity, R and β the distance and angle of projection, we have

$$V^2=\frac{2\mu}{n-1}\left(\frac{1}{R}\right)^{n-1}, \qquad h=VR\sin\beta\ \ldots\ldots\ldots\ldots (2).$$

Representing $\dfrac{2\mu}{h^2(n-1)}=\dfrac{R^{n-3}}{\sin^2\beta}$ by c^{n-3}, we have

$$\frac{du}{u\sqrt{\{(cu)^{n-3}-1\}}}=\mp d\theta \ldots\ldots\ldots\ldots (3),$$

where the upper or lower sign is to be taken according as $du/d\theta$ is initially negative or positive, i.e. according as the angle β is acute or obtuse.

To integrate this put $cu=x^\kappa$ where κ is to be chosen to suit our convenience. Taking the logarithmic differential we find $du/u=\kappa\,dx/x$, and the integral equation (3) becomes

$$\frac{\kappa\,dx}{x\sqrt{(x^{\kappa(n-3)}-1)}}=\mp d\theta.$$

We now see that if we put $\kappa(n-3)=-2$ the integration can be effected at once, but this supposition is impossible if $n=3$. We find

$$\kappa\cos^{-1}x=\pm(\theta-\alpha),\qquad \therefore \left(\frac{r}{c}\right)^{\frac{n-3}{2}}=\cos\frac{n-3}{2}(\theta-\alpha).$$

Conversely, when the path is given, we have

$$h^2=\frac{2\mu}{n-1}\frac{1}{c^{n-3}},\qquad v^2=\frac{2\mu}{n-1}\frac{1}{r^{n-1}}.$$

It appears that the orbit takes different forms according as $n>$ or <3. In the former case the curve has a series of loops with the origin for the common node and $r=c$ for the maximum radius vector. In the latter case the curve has infinite branches, and $r=c$ for the minimum radius vector.

361. If the force is repulsive, we write $F=-\mu' u^n$. We then have

$$v^2=h^2\left\{\left(\frac{du}{d\theta}\right)^2+u^2\right\}=\frac{2\mu'}{1-n}r^{1-n}+C.$$

If $C=0$, we must have $n<1$. *The velocity at every point is equal to that from rest at the centre of force.* Proceeding as before, we have

$$\left(\frac{c}{r}\right)^{\frac{3-n}{2}}=\cos\frac{3-n}{2}(\theta-\alpha).$$

362. *Ex.* The law of attraction being $F=\mu u^n$, show that the time t of describing a loop is

$$t\sqrt{\frac{\mu}{2(n-1)c^{n+1}}}=\int\left(\cos\frac{n-3}{2}\theta\right)^{\frac{4}{n-3}}d\theta=\frac{\sqrt{\pi}}{n-3}\frac{\Gamma(p)}{\Gamma(q)},$$

where the limits are $\theta=0$ to $\pi/(n-3)$ and $2(n-3)p=n+1$, $(n-3)q=(n-1)$. The integrations can be effected when $n-3=\pm 4/i$ and $q-p=\pm i$ where i is any integer.

363. Examples. *Ex.* 1. Prove the following geometrical properties of the curve $(r/c)^m=\cos m\theta$ (Art. 320),

$$\phi=\frac{\pi}{2}+m\theta,\qquad p=\frac{r^{m+1}}{c^m},\qquad \left(\frac{r'}{c}\right)^{\frac{m}{m+1}}=\cos\frac{m\theta'}{m+1},$$

where ϕ is the angle the radius vector makes with the tangent, and r', θ' are the coordinates of a point on the pedal curve.

Since equation (1) of Art. 360 becomes $p^2=\dfrac{(n-1)h^2}{2\mu}r^{n-1}$ when $C=0$, the second of these geometrical results enables us to write down the equation of the required path and thus to avoid the integration of (3).

Ex. 2. A perpendicular OY is drawn from the origin O on the tangent at P to the lemniscate $r^2=a^2\cos 2\theta$. If the locus of Y be described by a particle under the action of a central force tending to O, prove that this force varies inversely as $OY^{13/3}$. [Coll. Ex.]

Ex. 3. A particle is describing the curve $(r/c)^m=\cos m\theta$ under the action of the central force $F=\mu u^n$, where $m=\frac{1}{2}(n-3)$. Prove that, if the velocity at the

point $\theta=\alpha$ is suddenly increased in the ratio 1 to $1+\gamma$ where γ is very small, the subsequent path is

$$(r/c)^m=\cos m\theta\,\{1-m\xi\,(\cos m\theta)^{\frac{1}{m}}\},$$

$$\frac{(\cos m\theta)^{1+\frac{1}{m}}}{(\sin m\theta)}\,\xi=\gamma\,\frac{\cot m\theta}{m}+\frac{\gamma}{(\cos m\alpha)^{\frac{2m+2}{m}}}\int\frac{(\cos m\theta)^{2+\frac{2}{m}}\,d\theta}{(\sin m\theta)^2},$$

where the limits are $\theta=0$ to α.

Substitute $r/c=(\cos m\theta)^{\frac{1}{m}}+\xi$, in the differential equation of the path, Art. 309, and neglect the squares of ξ.

364. The inverse fifth power. The equation (1), Art. 360, has the form

$$\left(\frac{du}{d\theta}\right)^2=\frac{\mu}{2h^2}u^4-u^2+\frac{C}{h^2}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(1).$$

This can be reduced to elliptic integrals as explained in Cayley's *Elliptic Functions*, Art. 400, or Greenhill, *The Elliptic Functions*, Art. 70.

The integration can be effected in two cases: (1) when velocity of projection is equal to that from infinity, and (2) when the initial conditions are such that $h^4=2\mu C$. In the latter case the right-hand side of (1) is a perfect square.

Ex. 1. Prove that the integration when $h^4=2\mu C$ leads to the curves $\tanh(\theta/\sqrt{2})=r/c$ or c/r, which have a common asymptotic circle $r=c$ where $c=\sqrt{\mu}/h$. Prove also that the velocity V of projection is given (Art. 313) by

$$V^2\sin^4\beta=2V'^2\{1\pm\sqrt{(1-\sin^4\beta)}\},$$

where V' is the velocity from rest at infinity, and the upper or lower sign is to be taken according as the path is outside or inside the asymptotic circle.

Ex. 2. Prove that, if the central force $F=\mu u^5$, the inverse of any path with regard to the origin is another possible path provided the total energy of the motion exceed the potential energy at infinity by a positive constant E reckoned per unit mass and also that for the two paths $Eh'^4=E'h^4$.

Prove that when $h^4>4\mu E>0$ the path is of the form $r=a\,\mathrm{sn}\left(K-\dfrac{\theta}{\sqrt{(1+k^2)}}\right)$ modulus k or the inverse form. [Math. Tripos, 1894.]

According to the notation of Art. 313, $2E=C$.

365. The inverse fourth power. The equation (1) of Art. 360 is

$$\left(\frac{du}{d\theta}\right)^2=\frac{2\mu}{3h^2}\left(u^3-\frac{3h^2}{2\mu}u^2+\frac{3C}{2\mu}\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(1).$$

This cubic can always be written in the form

$$\left(\frac{du}{d\theta}\right)^2=\frac{2\mu}{3h^2}(u+a)\,(u^2+Au+B),$$

and the integration can be reduced to forms similar to those in Art. 364 by writing $u+a=\xi^2$.

The integration can be effected when the initial conditions are such that $h^6=3\mu^2C$. In this case the right-hand side has the factor $(u-h^2/\mu)^2$.

Ex. Show that the integration leads to the curves $u=\dfrac{h^2}{\mu}\dfrac{\cosh\theta\pm2}{\cosh\theta\mp1}$, the upper signs being taken together and the lower together. These curves have a common asymptotic circle $r=\mu/h^2$, one curve being within and the other outside.

366. Other powers. *Ex.* If the force $F=\mu u^7$, and the initial conditions are such that $2h^3=3C\sqrt{\mu}$, prove that the equation (1) of Art. 360 takes the form

$$\left(\frac{du}{d\theta}\right)^2=\frac{1}{3b^4}(u^2-b^2)^2(u^2+2b^2),$$

where $b^2=h/\sqrt{\mu}$. Thence deduce the integrals $\dfrac{u^2}{b^2}=\dfrac{\cosh 2\theta\mp 1}{\cosh 2\theta\pm 2}$, having a common asymptotic circle. The Lemniscate can also be described under this law of force, if the velocity is equal to that from infinity; Arts. 320, 360.

367. Nearly circular orbits. *To find the motion approximately, when the central force $F=\mu u^n$ and the orbit is nearly circular.*

Beginning as in Art. 360 with the equation

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}=\frac{\mu}{h^2}u^{n-2}\text{ ....(1)},$$

we put $u=c(1+x)$ where c is some constant to be presently chosen but subject to the condition that x is to be a small fraction. We thus find

$$\frac{d^2x}{d\theta^2}+x=-1+\frac{\mu c^{n-3}}{h^2}\left\{1+(n-2)x+\frac{(n-2)(n-3)}{2}x^2+\&c.\right\}\text{ (2)}.$$

We see now that the right-hand side of the equation will be simplified if we choose c so that the constant term is zero, i.e. we put $h^2=\mu c^{n-3}$. The equation then becomes

$$\frac{d^2x}{d\theta^2}+x=(n-2)x+\tfrac{1}{2}(n-2)(n-3)x^2+\&c.\text{ ......(3)}.$$

As a first approximation, we assume

$$x=M\cos(p\theta+\alpha)\text{......................(4)},$$

where M is a small quantity. Substituting and rejecting the squares of M we find

$$(1-p^2)M\cos(p\theta+\alpha)=(n-2)M\cos(p\theta+\alpha)\text{......(5)}.$$

The differential equation is therefore satisfied to the first order, if we put $p^2=3-n$. In this case we have as the equation of the path

$$u=c\{1+M\cos(p\theta+\alpha)\}\text{ .. ..(6)}.$$

If $n<3$, the equation (6) represents a real first approximate solution of the differential equation (1). We notice that the particle oscillates between the two circles $u=c(1+M)$ and $u=c(1-M)$. The meaning of the constant c is now apparent; geometrically, $1/c$ is the harmonic mean of the radii of the bounding circles; dynamically, $1/c$ is the radius of that circle which

would be described about the centre of force with the given angular momentum h.

The positions of the apses are found by equating $du/d\theta$ to zero. This gives $p\theta+\alpha=i\pi$, the angle at the centre of force between two successive apses is therefore π/p.

If $n>3$, the value of p is imaginary, and the trigonometrical expression takes a real exponential form, Art. 120. The quantity x therefore becomes large when θ increases, and the particle, instead of remaining in the immediate neighbourhood of the circumference of the circle, deviates widely from it on one side or the other. As the square of x has been neglected the exponential form of (6) only gives *the initial stage of the motion* and ceases to be correct when x has become so large that its square cannot be neglected. It follows from this that *the motion of a particle in a circle about a centre of force in the centre is unstable if* $n>3$.

368. *Ex.* If the law of force is $F=u^2f(u)$, and the orbit is nearly circular, prove that a first approximation to the path is

$$u=c\{1+M\cos(p\theta+\alpha)\},\qquad p^2=1-\frac{cf'(c)}{f(c)}.$$

Thence it follows that *the apsidal angle is independent of the mean reciprocal radius*, viz. c, only when $F=\mu u^n$, i.e., *when the law of force is some power of the distance.*

369. A second approximation. The solution (6) is in any case only a first approximation to the motion, and it may happen that, when we proceed to a second or third approximation, the value of p is altered by terms which contain M as a factor. Besides this, we shall have x expressed in a series of several trigonometrical terms whose general form is $N\cos(q\theta+\beta)$, where N contains the square or cube of M as a factor together with some divisor κ introduced by the integration, Arts. 139, 303.

Representing the corrected value of p by $p+\Delta$, the error in $p\theta+\alpha$, i.e. $\theta\Delta$, increases by $2\pi\Delta$ after each successive revolution of the particle round the centre of force. The expression (6) will therefore cease to be even a first approximation as soon as $\theta\Delta$ has become too large to be neglected. On the other hand the additional term to the value of u may be comparatively unimportant. The magnitude of the specimen term is never greater than N and, unless κ is also small, we can generally neglect such terms.

In proceeding to a higher approximation we should first seek for those terms in the differential equation which contain $\cos(p\theta+\alpha)$; these being added to the terms of the same form in equation (5) will modify the first approximate value of p.

We should also enquire if any term in the differential equation acquires by integration a small divisor κ and thus becomes comparatively large in the solution.

370. To obtain a second approximation we substitute the first approximation (6) in the *small* terms of the differential equation (3). Writing (3), for brevity, in the form

$$\frac{d^2x}{d\theta^2}=(n-3)\{x+\beta x^2+\gamma x^3+\dots\} \qquad (7),$$

where $\beta=\frac{1}{2}(n-2)$, $\gamma=\frac{1}{6}(n-2)(n-4)$, &c., we find after rejecting the cubes of M

$$\frac{d^2x}{d\theta^2}=(n-3)\{x+\tfrac{1}{2}\beta M^2(1+\cos 2p\theta)\} \qquad (8),$$

where $p\theta$ has been written for $p\theta+\alpha$ for the sake of brevity. This equation shows (Art. 303) that the second approximate value of x has the form

$$x=M\cos p\theta+M^2(G+A\cos 2p\theta) \qquad (9),$$

where G and A are two constants whose values may be found by substitution, and p has the same value as before.

To obtain a third approximation, we retain the term γx^3 in (7) and assume

$$x=M\cos p\theta+M^2(G+A\cos 2p\theta)+M^3B\cos 3p\theta \qquad (10).$$

To find the values of p, G, A and B we substitute in (7), express all the powers of the trigonometrical functions in multiple angles and neglect all terms of the order M^4. Equating the coefficients of $\cos p\theta$, $\cos 2p\theta$, $\cos 3p\theta$ and the constants on each side, we find

$$\begin{aligned}
-Mp^2&=(n-3)\{M+2M^3G\beta+M^3A\beta+\tfrac{3}{4}M^3\gamma\},\\
-4M^2p^2A&=(n-3)\{M^2A+\tfrac{1}{2}M^2\beta\},\\
-9M^3p^2B&=(n-3)\{M^3B+M^3A\beta+\tfrac{1}{4}M^3\gamma\},\\
0&=M^2G+\tfrac{1}{2}M^2\beta.
\end{aligned}$$

Solving these equations, and remembering that p^2 differs from $3-n$ by terms of the order M^2, we find

$$G=-\tfrac{1}{4}(n-2),\qquad A=\tfrac{1}{12}(n-2),\qquad B=\tfrac{1}{96}(n-2)(n-3),$$

$$p^2=(3-n)\{1-\tfrac{1}{12}(n-2)(n+1)M^2\} \qquad (11).$$

The three first are correct when M^2 is neglected and the last when M^4 is neglected.

We notice that up to and including the third order of approximation the terms G, A, B in equation (10) do not contain any small denominators, so that if M be small enough all these terms may be neglected. The motion is then represented very nearly by

$$u=c\{1+M\cos(p\theta+\alpha)\} \qquad (12),$$

$$p=\sqrt{(3-n)}\{1-\tfrac{1}{24}(n-2)(n+1)M^2\} \qquad (13),$$

and this approximation holds until θ gets so large that $M^4\theta$ cannot be neglected. We notice also that *the additional term in the value of p vanishes only when the law of force is either the inverse square or the direct distance.*

Disturbed Elliptic Motion.

371. Impulsive disturbance. When a particle is describing an orbit about a centre of force it may happen that at some particular point of that orbit the particle receives an impulse and begins to describe another orbit. We have to determine

how the new orbit differs from the old, for example how the major axis has been changed in position and magnitude, and in general to express the elements of the new orbit in terms of those of the undisturbed orbit.

Let the unaccented letters a, e, l, &c. represent the elements of the undisturbed orbit, while the accented letters a', e', l', &c. represent corresponding quantities for the new. We first express the velocity v and the angle β at the given point of the orbit in terms of the undisturbed elements. Thus v and β are given by

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right), \quad \sin\beta = \frac{p}{r} = \frac{\sqrt{\mu l}}{vr} \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

when the undisturbed orbit is an ellipse described about the focus.

We next consider the circumstances of the blow. Let m be the mass of the particle, mB the blow. The particle, after the impulse is concluded, is animated with the velocity B in the given direction of the blow, together with the velocity v along the tangent to the original path. Compounding these the particle has a resultant velocity v' and is moving in a known direction. Since the position of the radius vector is not changed by the blow we may conveniently refer the changes of motion to that line. If P, Q are the components of B along and perpendicular to the radius vector and β' is the angle the direction of motion makes with the radius vector, we have

$$v'\cos\beta' = v\cos\beta + P, \quad v'\sin\beta' = v\sin\beta + Q \;\ldots\ldots (2).$$

Having now obtained v', β', the formulæ (1), writing accented letters for the old elements, determine the new semi-major axis a' and the new semi-latus rectum l'. The position in space of the major axis follows from Art. 336.

372. We may sometimes advantageously replace the second of the equations (1) by another formula. We notice that mh is the moment of the momentum of the particle about the centre of force. Since just after the impulse the velocity v' is the resultant of v and B, *the moment of v' is equal to that of v together with the moment of B.* Hence

$$h' = h + Bq \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where q is the perpendicular on the line of action of the blow. Since $h^2 = \mu l$, when the law of force follows the Newtonian law,

this equation leads to

$$\sqrt{l'} = \sqrt{l} + Bq/\sqrt{\mu} \quad\text{.....................(4).}$$

Thus the change in the latus rectum is very easily found.

As a corollary, we may notice that *when the blow acts along the radius vector, the angular momentum mh and therefore the latus rectum of the orbit are unchanged.* We also observe that if the magnitude of the attracting force or its law of action were abruptly changed, the value of h is unaltered.

373. *Ex.* 1. Two particles, describing orbits about the same centre of force, impinge on each other. Prove

$$m_1h_1' + m_2h_2' = m_1h_1 + m_2h_2,$$

where m_1h_1, m_2h_2; m_1h_1', m_2h_2' are their angular momenta before and after impact.

Ex. 2. A particle P of unit mass is describing an ellipse about the focus S. A circle is described to touch the normal to the conic at P whose radius PC represents the velocity at P in direction and magnitude. Prove that if the particle is acted on by an impulse represented in direction and magnitude by any chord MP of the circle, the length of the major axis is unaltered by the blow.

Since $B = 2v\cos\theta$, the velocity in the direction of the blow is simply reversed. Hence $v' = v$ and $a' = a$ by Art. 335.

374. If the direction of the blow does not lie in the plane of motion, the plane of the new orbit is also changed. For the sake of the perspective, let the radius vector SP be the axis of x and let the plane of xy be the plane of the old orbit; then $v\cos\beta$, $v\sin\beta$ are the components of velocity parallel to the axes of x and y. Let the components of the blow be mX, mY, mZ; then just after the blow is concluded the components of velocity parallel to the axes are $v\cos\beta + X$, $v\sin\beta + Y$, and Z. The inclination i of the planes of the two orbits is therefore given by $\tan i = \dfrac{Z}{v\sin\beta + Y}$. The particle begins to move in its new orbit with a velocity v' in a direction making an angle β' with the radius vector SP given by

$$v'\cos\beta' = v\cos\beta + X, \qquad (v'\sin\beta')^2 = (v\sin\beta + Y)^2 + Z^2.$$

The problem is now reduced to the case already considered.

If mh' is the angular momentum in the new orbit, its components about the axes of x, y, z are 0, $-mh'\sin i$, $mh'\cos i$. Hence

$$h'\cos i = h + Yr, \qquad h'\sin i = Zr,$$

where $r = SP$.

375. Examples. *Ex.* 1. A particle is describing a given ellipse about a centre of force in the focus, and when at the farther apse A', its velocity is suddenly increased in the ratio $1 : n$. Find the changes in the elements.

The direction of motion is unaltered by the blow and since this direction is at right angles to the radius vector from the centre of force, the point A' is one of the apses of the new orbit.

Let a, e; a', e' be the semi-major axes and eccentricities of the orbits. Then since SA' is unaltered in length

$$r = a'(1 + e') = a(1 + e) \quad\text{..................................(1).}$$

We have here chosen as the standard figure for the new orbit an ellipse having A' for the further apse. A negative value of the eccentricity e' therefore means that A' is the nearer apse.

Also since $v'=nv$, we have

$$\mu\left(\frac{2}{r}-\frac{1}{a'}\right)=n^2\mu\left(\frac{2}{r}-\frac{1}{a}\right) \quad\text{.......... ....} \quad (2),$$

where a' must be regarded as negative if the new orbit is a hyperbola, Art. 333.

From these equations we find

$$\frac{a'}{a}=\frac{1+e}{2-n^2(1-e)}, \qquad e'=1-n^2(1-e).$$

The point A' is therefore the farther or nearer apse according as $n^2(1-e)$ is $<$ or >1; if equal to unity the new orbit is a circle, if equal to -1, a parabola. The new orbit is an ellipse or hyperbola according as $n^2(1-e) <$ or >2.

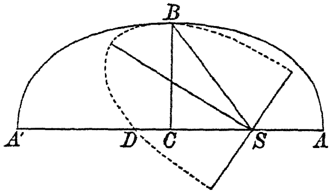
Ex. 2. A particle describes an ellipse under a force tending to a focus. On arriving at the extremity of the minor axis, the force has its law changed, so that it varies as the distance, the magnitude at that point remaining the same. Prove that the periodic time is unaltered and that the sum of the new axes is to their difference as the sum of the old axes to the distance between the foci.

[Math. Tripos, 1860.]

By Art. 325 the new orbit is an ellipse having the centre of force S in the centre. Let the new law of force be $\mu' r$. Then when $r=a$, the forces are equal, hence

$$\mu' a=\mu/a^2 \text{ ....} \quad (1).$$

Measure a length SD parallel to the direction of motion at B, such that the velocity v at B is $\sqrt{\mu'}\,.\,SD$. Then SD is the semi-conjugate of SB in the new orbit. Equating the velocities at B in the old and new orbits, we have when $r=a$



$$\mu\left(\frac{2}{r}-\frac{1}{a}\right)=\mu'\,.\,SD^2, \qquad \therefore\ SD=a \text{ ....} \quad (2).$$

The conjugates SB, SD are equal diameters, the major and minor axes are therefore the internal and external bisectors of the angle BSD. Representing the semi-axes by a', b', we have

$$a'^2+b'^2=SB^2+SD^2=2a^2, \qquad a'b'=SB\,.\,SD\sin BSD=ab \text{ ....} \quad (3).$$

The internal bisector of the angle BSD is clearly the major axis.

If the change in the velocity had been made at any point of the ellipse, we proceed in the same way. By drawing SD parallel to the direction of motion we arrive at the known problem in conics, given two conjugate diameters in position and magnitude, construct the ellipse.

The periodic times in the two orbits are respectively $2\pi/\sqrt{\mu'}$ and $2\pi\sqrt{a^3/\mu}$. The equality of these follows from the equation (1). The rest of the question follows from (3).

Ex. 3. A particle is describing an ellipse under a force μ/r^2 to a focus: when the particle is at the extremity of the latus rectum through the focus this centre of force is removed and is replaced by a force $\mu' r'$ at the centre of the ellipse. Prove that if the particle continue to describe the same ellipse $\mu' b^4=\mu a$.

[Coll. Exam. 1895.]

Ex. 4. A planet moving round the sun in an ellipse receives at a point of its orbit a sudden velocity in the direction of the normal outwards which transforms the orbit into a parabola, prove that this added velocity is the same for all points of the orbit, and if it be added at the end of the minor axis, the axis of the parabola will make with the major axis of the ellipse an angle whose sine is equal to the eccentricity. [Coll. Exam. 1892.]

Ex. 5. A particle describes a given ellipse about a centre of force of given intensity in the focus S. Supposing the particle to start from the further extremity of the major axis, find the time T of arriving at the extremity of the minor axis. At the end of this time the centre of force is transferred without altering its intensity from S to the other focus H, and the particle moves for a second interval T equal to the former under the influence of the central force in H. Find the position of the particle, and show that, if the centre of force were then transferred back to its original position, the particle would begin to describe an ellipse whose eccentricity is $(3e-e^2)/(1+e)$. [Math. Tripos, 1893.]

Ex. 6. A body is describing an ellipse round a force in its focus S, and HZ is the perpendicular on the tangent to the path from the other focus H. When the body is at its mean distance the intensity of the force is doubled, show that SZ is the new line of apses. [Coll. Ex.]

Ex. 7. A particle describes a circle of radius c about a centre of force situated at a point O on the circumference. When P is at the distance of a quadrant from O, the force without altering its instantaneous magnitude begins to vary as the inverse square. Prove that the semi-axes of the new orbit are $\frac{2}{3}c\sqrt{2}$ and $\frac{1}{3}c\sqrt{3}$.

Ex. 8. Two inelastic particles of masses m_1, m_2, describing ellipses in the same plane impinge on each other at a distance r from the centre of force. If $a_1, l_1; a_2, l_2$; are the semi-major axes and semi-latera recta before impact, prove that in the ellipse described after impact

$$(m_1+m_2)\,l^{\frac{1}{2}}=m_1l_1^{\frac{1}{2}}+m_2l_2^{\frac{1}{2}},$$

$$(m_1+m_2)\left(2r-l-\frac{r^2}{a}\right)^{\frac{1}{2}}=m_1\left(2r-l_1-\frac{r^2}{a_1}\right)^{\frac{1}{2}}+m_2\left(2r-l_2-\frac{r^2}{a_2}\right)^{\frac{1}{2}}.$$

Ex. 9. A planet, mass M, revolving in a circular orbit of radius a, is struck by a comet, mass m, approaching its perihelion; the directions of motion of the comet and planet being inclined at an angle of 60°. The bodies coalesce and proceed to describe an ellipse whose semi-major axis is $\dfrac{(M+m)^2a}{M\{M+(4-\sqrt{2})m\}}$. Prove that the original orbit of the comet was a parabola; and if the ratio of m to M is small, show that the eccentricity of the new orbit is $(7\frac{1}{2}-4\sqrt{2})^{\frac{1}{2}}(m/M)$.

[Coll. Ex. 1895.]

376. Continuous forces. We may apply the method of Art. 371 to find the effects of continuous forces on the particle. Let f, g be the tangential and normal accelerating components of any disturbing force, the first being taken positively when increasing the velocity and the second when acting inwards.

We divide the time into intervals each equal to δt and consider

the effect of the forces on the elements of the ellipse at the end of each interval. We treat the forces, in Newton's manner, as small impulses generating velocities $f\delta t$ and $g\delta t$ along the tangent and normal respectively. The effect of the tangential force is to increase the velocity at any point P from v to $v+\delta v$, where $\delta v = f\delta t$, the direction of motion not being altered. To find the effect of the normal force we observe that after the interval δt the particle has a velocity $g\delta t$ along the normal, while the velocity v along the tangent is not altered. The direction of motion has therefore been turned round through an angle $\delta\beta = g\delta t/v$.

If the disturbing force were now to cease to act, the particle would move in a conic whose elements could be deduced from these two facts, (1) the velocity at P is changed to $v+\delta v$, (2) the angle of projection is $\beta+\delta\beta$. *The conic which the particle would describe if at any instant the disturbing forces were to cease to act is called the instantaneous conic at that instant.*

377. *To find the effect on the major axis,* we use the formula

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Since v is increased to $v+\delta v$, we see by simple differentiation

$$2v\,\delta v = \frac{\mu}{a^2}\,\delta a, \qquad \therefore\ \delta a = \frac{2a^2 v}{\mu} f\delta t \quad\ldots\ldots\ldots\ldots(2).$$

In differentiating the formula for v^2 we are not to suppose that δv represents the whole change of the velocity in the time δt. The particle moves along the ellipse and experiences a change of velocity dv in the time dt given by

$$v\,dv = -\frac{\mu}{r^2}\,dr \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Taking $dt=\delta t$, the change of velocity in the time δt is $\delta v + dv$, the part δv being due to the disturbing forces and the part dv to the action of the central force.

378. *To find the changes in the eccentricity and line of apses.* We may effect this by differentiating the formulæ

$$l = a(1-e^2), \quad h^2 = \mu l, \quad \frac{l}{r} = 1 + e\cos\theta \quad\ldots\ldots\ldots\ldots(4).$$

Since mh is the angular momentum, the increase of mh, viz. $m\delta h$, is equal to the moment of the disturbing forces about the origin (Art. 372). Let β be the angle the direction of motion at P makes with the radius vector,

$$\therefore\ \tfrac{1}{2}\sqrt{\mu}\,\frac{\delta l}{\sqrt{l}} = \delta h = fr\sin\beta + gr\cos\beta.$$

We deduce from equations (4)

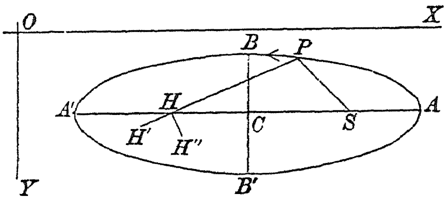
$$\delta l = (1-e^2)\,\delta a - 2ae\delta e, \quad \frac{\delta l}{r} = \cos\theta\,\delta e - e\sin\theta\,\delta\theta,$$

and the values of δe and $\delta\theta$ follow at once.

379. Herschel has suggested a geometrical method of finding the changes of the eccentricity and the line of apses in his *Outlines of Astronomy**. He considers the effect of the disturbing forces f, g on the position of the empty focus.

The effect of the tangential force f is to alter the velocity v and therefore to alter a. Since $SP+PH=2a$, the empty focus H is moved, during each interval δt, along the straight line PH a distance $HH'=2\delta a$, where δa is given by (2).

The effect of the normal force g is to turn the tangent at P through an angle $\delta\beta = g\delta t/v$. Since SP, HP make equal angles with the tangent, the empty focus H is moved perpendicularly to PH, a distance $HH''=2PH\,.\,\delta\beta$.



Consider first the tangential force f, we have $SH=2ae$, $SH'=2(ae+\delta ae)$. Hence projecting on the major axis

$$2\delta(ae) = HH'\,.\,\cos PHS = 2\delta a\,\frac{x+ae}{r'},$$

where $r'=HP=a+ex$, and x is measured from the centre;

$$\therefore\ \delta e = \frac{x-e^2x}{r'}\,\frac{\delta a}{a} = \frac{2a(1-e^2)}{\mu}\,\frac{xv}{r'}\,f\delta t.$$

Let ϖ be the longitude of the apse line HS measured from some fixed line through S,

$$\therefore\ 2ae\delta\varpi = HH'\sin PHS = 2\delta a\,\frac{y}{r'},$$

$$\therefore\ e\delta\varpi = \frac{y}{r'}\,\frac{\delta a}{a} = \frac{2a}{\mu}\,\frac{yv}{r'}\,f\delta t.$$

Consider secondly the normal force g. We have

$$SH=2ae. \quad SH''=2(ae+\delta ae), \quad \delta a = 0;$$

$$\left.\begin{aligned} \therefore\ 2\delta(ae) &= -HH''\sin PHS = -2r'\delta\beta\,\frac{y}{r'} \\ 2ae\delta\varpi &= \frac{HH''\cos PHS}{r'} = 2r'\delta\beta\,\frac{x+ae}{r'} \end{aligned}\right\};$$

$$\therefore\ \delta e = -\frac{1}{a}\,\frac{y}{v}\,g\delta t, \quad e\delta\varpi = \frac{1}{a}\,\frac{x+ae}{v}\,g\delta t.$$

* See also some remarks by the author in the *Quarterly Journal*,
It should be noticed that Herschel measures the eccentricity by half the distance between the foci, a change from the ordinary definition which has not been followed

380. The expressions for δe, $\delta\varpi$ should be put into different forms according to the use we intend to make of them. Let ψ be the angle the tangent at P makes with the major axis, then $\tan\psi = \frac{b^2x}{a^2y}$. We easily find by elementary conics

$$\sin\psi = \frac{b}{a}\frac{x}{\sqrt{(rr')}}, \qquad \cos\psi = \frac{a}{b}\frac{y}{\sqrt{(rr')}}.$$

Also $v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) = \frac{\mu r'}{ar}$. It immediately follows that

$$\delta e = \frac{2b}{\sqrt{(\mu a)}} f \sin\psi\,\delta t, \qquad e\,\delta\varpi = \frac{2b}{\sqrt{(\mu a)}} f\cos\psi\,\delta t,$$

$$\delta e = -\frac{br}{a\sqrt{(\mu a)}} g\cos\psi\,\delta t, \qquad e\,\delta\varpi = \frac{ar}{b\sqrt{(\mu a)}}\,\frac{x+ae}{x}\, g\sin\psi\,\delta t.$$

These formulæ give the changes of e and ϖ produced by any tangential or normal force.

381. Draw two straight lines OX, OY parallel to the principal diameters situated as shown in the figure. Since $f\cos\psi$, $f\sin\psi$ are the components of the tangential disturbing force parallel to the principal diameters, we see that *when the force acts towards OX the eccentricity is increased, and when towards OY the apse line is advanced*; the contrary effects taking place when the force tends from these lines.

The same rule applies to the normal disturbing force so far as the eccentricity is concerned. It applies also to the motion of the apse except when the particle lies between the minor axis and the latus rectum through the empty focus, and the rule is then reversed. When the eccentricity is small, $\frac{e\sin\psi}{x} = \frac{be}{a^2}$ very nearly when the particle is near the minor axis; so that the effects of the tangential force in this part of the orbit may be neglected and the rule applied generally.

382. Examples. *Ex.* 1. The path of a comet is within the orbit of Jupiter, approaching it at the aphelion. Show that each time the comet comes near Jupiter the apse line is advanced. This theorem is due to Callandreau, 1892.

The comet being near the aphelion and Jupiter just beyond, both the normal and tangential disturbing forces act towards OY; the apse therefore advances.

Ex. 2. A particle is describing an elliptic orbit about the focus and at a certain point the velocity is increased by $1/n$th, n being large. Prove that, if the direction of the major axis be unaltered, the point must be at an apse, and the change in the eccentricity is $2(1 \pm e)/n$. [Coll. Ex. 1897.]

Ex. 3. An ellipse of eccentricity e and latus rectum l is described freely about the focus by a particle of mass m, the angular momentum being mh. A small impulse mu is given to the particle, when at P, in the direction of its motion; prove that the apsidal line is turned through an angle which is proportional to the intercept made by the auxiliary circle of the ellipse on the tangent at P, and which cannot exceed lu/eh. [Math. Tripos, 1893.]

Ex. 4. A body describes an ellipse about a centre of force S in the focus. If A be the nearer apse, P the body, and a small impulse which generates a velocity T act on the body at right angles to SP, prove that the change of direction of the

apse line is given approximately by $\frac{T}{h}\left(\frac{2}{e}+\cos ASP\right) SP \sin ASP$, where e is the eccentricity of the orbit and h twice the rate of description of area about S.

[Math. Tripos.]

Ex. 5. A particle describes an ellipse about a centre of force in the focus S. When the particle has reached any position P the centre of force is suddenly moved parallel to the tangent at P through a short distance x, prove that the major axis of the orbit is turned through the angle $\frac{x}{SG}\sin\phi\sin(\theta-\phi)$ where G is the point at which the normal at P meets the original major axis, θ the angle SGP and ϕ the angle the tangent makes with SP. [Coll. Ex. 1895.]

Ex. 6. A particle describes an ellipse about a centre of force μ/r^2 and is besides acted on by a disturbing force κr^n tending to the same point. Prove that as the particle moves from a distance r_0 to r, the major axis and eccentricity change according to the law

$$\mu\left(\frac{1}{a}-\frac{1}{a_0}\right)=\frac{2\kappa}{n+1}(r^{n+1}-r_0{}^{n+1}),\qquad \frac{1-e^2}{1-e_0{}^2}=\frac{a_0}{a}.$$

Thence deduce the changes in a and e when κ is very small.

383. A resisting medium. We may also use the formulæ of Art. 380 to find the quantitative effect of a resisting medium on the motion of a particle describing an ellipse about a centre of force in the focus.

The velocity of the particle being v, let the resistance be κv. Then $g=0$ and $f=-\kappa ds/dt$, and the equations of motion become

$$\frac{de}{dt}=-\frac{2b\kappa}{\sqrt{(\mu a)}}\frac{dy}{dt},\qquad e\frac{d\varpi}{dt}=-\frac{2b\kappa}{\sqrt{(\mu a)}}\frac{dx}{dt}.$$

Usually f and g are so small that their squares can be neglected. Now the changes of the elements a, e, &c. are of the order of f and g, being produced by these forces. Hence in using these equations *we may regard the elements of the ellipse, when multiplied by the coefficient κ of resistance, as constants.*

Supposing then that we reject the squares of κ, we have by an easy integration

$$e=-\frac{2b\kappa}{\sqrt{(\mu a)}}y+A,\qquad e\varpi=-\frac{2b\kappa}{\sqrt{(\mu a)}}x+B,$$

where A, B are two undetermined constants. Since after a complete revolution, the coordinates x, y return to their original values, both the eccentricity and the position of the line of apses must also be the same as before. *There can therefore be no permanent change in either.* The greatest change of the eccentricity from

its mean value is $2\kappa b^2/na^2$, while the apse oscillates about its mean position through an angle $2\kappa b/nea$, where $\mu = n^2a^3$, Art. 341.

384. *Ex.* A comet moves in a resisting medium whose resistance is $f = -\kappa V^p \left(\frac{a}{r}\right)^q$ where V is the velocity, r the distance from the sun and p, q are positive quantities. When the true anomaly θ is taken as the independent variable (instead of t as in Art. 380), prove that

$$\frac{1}{a}\frac{da}{d\theta} = \frac{-2A}{1-e^2}(1+2e\cos\theta+e^2)^{\frac{p+1}{2}}(1+e\cos\theta)^{q-2},$$

$$\frac{de}{d\theta} = -2A(\cos\theta+e)(1+2e\cos\theta+e^2)^{\frac{p-1}{2}}(1+e\cos\theta)^{q-2},$$

$$e\frac{d\varpi}{d\theta} = -2A\sin\theta(1+2e\cos\theta+e^2)^{\frac{p-1}{2}}(1+e\cos\theta)^{q-2},$$

where $$A = \kappa n^{p-2}a^{p-1}.(1-e^2)^{2-\frac{p}{2}-q} \text{ and } \mu = n^2a^3.$$

When the right-hand sides of these equations are expanded in series of the form

$$A + B\cos\theta + C\cos 2\theta + \ldots$$

it is obvious that the only permanent changes are derived from the non-periodical terms. Prove (1) that the longitude of the apse has no permanent changes, (2) that the eccentricity at the time t is $e - Aent(p+q-1)$, (3) the semi-major axis is $a - 2Aant$. These results are given by Tisserand, *Méc. Céleste*, 1896.

When the law of resistance is such that $p+q=1$, it follows that *neither the eccentricity nor the line of apses have any permanent change.* For any values of p and q not satisfying this relation the eccentricity will gradually change and continue to change in the same direction. When the changes of any of the elements have become so great that their products by the coefficient κ of resistance can no longer be neglected, the equations given above must be integrated in a different way.

385. Encke's Comet. The general effect of a resisting medium on the motion of a comet is to diminish its velocity and therefore also the major axis of its orbit, Art. 377. The ellipse which the comet describes is therefore continually growing smaller and the periodic time, which varies as $a^{3/2}$, continually decreases.

Encke was the first who thoroughly investigated the effect of a resisting medium on the motion of a comet. This comet has since then been called after his name. After making allowance for the disturbance due to the attraction of the sun and the planets, he found by observation that its period, viz. 1200 days, was diminished by about two hours and a half in each revolution. This he ascribed to the presence of a medium whose resistance varied as $(v/r)^2$ where v is the velocity of the comet and r its distance from the sun.

The importance and interest of Encke's result caused much attention to be given to this comet. The astronomers Von Asten of Pulkowa and afterwards Backlund* studied its motions at each successive appearance with the greatest

* In the *Bulletin Astronomique*, 1894, page 473, there is a short account of the work of Backlund by himself. He speaks of the continued decrease of the acceleration, the law of resistance, and gives references to his memoirs and particularly to

attention. The acceleration of the comet's mean motion appears to have been uniform from 1819, when Encke first took up the subject, to 1858. It then began to decrease and continued to decrease until the revolution of 1868—1871 when its magnitude was about half its former value. From 1871 to 1891 the acceleration was again nearly constant.

Assuming the law of resistance to be represented by $\kappa v^m/r^n$, Backlund found that n is essentially negative. This would make the density of the resisting medium increase according to a positive power of the distance from the sun; a result which he considered very improbable. He afterwards arrived at the conclusion that we must replace $1/r^n$ by some function $f(r)$ having maxima and minima at definite distances from the sun. In Laplace's nebular theory the planets are formed by condensations from rings of the solar nebula. In this formation all the substance of each ring would not be used up and some of it might travel along the orbit as a cloud of light material. It is suggested that Encke's comet passes through nebulous clouds of this kind and that the resistance they offer causes the observed acceleration.

It is known that comets contract on approaching the sun, sometimes to a very great extent. Tisserand remarks that when the size of the comet decreases the resistance should also decrease, and that this may help us to understand how the resistance to any comet might vary as a positive power of the distance from the sun. The size of Encke's comet also is not the same at every appearance and this again may have an effect on the law of resistance.

It is clear that if Encke's comet does meet with a resistance, every comet of short period which approaches closely to the sun must show the effect of the same influence. In 1880 Oppolzer thought he had discovered an acceleration in the motion of another comet. This was the comet Winnecke having a period of 2052 days. Further investigation showed that this was illusory, so that at present the evidence for the existence of a resisting medium rests on Encke's comet alone.

386. *Does the evidence afforded by Encke's comet prove a resisting medium?* Sir G. Stokes in a lecture* on the luminiferous medium says he asked the highest astronomical authority in the country this question. Prof. Adams replied that there might be attracting matter within the orbit of Mercury which would account for it in a different way. Sir G. Stokes then goes on to say that the comet throws out a tail near the sun and that this is equivalent to a reaction on the head towards

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the eighth volume of his *Calculs et Recherches sur la comète d'Encke.* In the *Comptes Rendus*, 1894, page 545, Callandreau gives a summary of the results of Backlund. In the *Traité de Mécanique Céleste*, vol. IV. 1896, Tisserand discusses the influence of a resisting medium. In the *History of Astronomy* by A. M. Clerke, 1885, examples of the contraction of comets near the sun are given. M. Valz in a letter to M. Arago quoted in the *Comptes Rendus*, vol. VIII. 1838, speaks of the great contraction of a comet as it approached the sun. He remarks that as it was approaching the earth at that time, it should have appeared larger. See also Newcombe's *Popular Astronomy*, 1883.

* Presidential address at the anniversary meeting of the *Victoria Institute*, June 29, 1893: reported in *Nature*, July 27, page 307.

the sun. There is therefore an additional force towards the sun. The effect of this would be to shorten the period even if there were no resisting medium. In the course of his lecture he discusses the question, "*must the ether retard a comet*," and decides that we cannot with safety infer that the motion of a solid through it necessarily implies resistance.

Kepler's Laws and the law of gravitation.

387. **Kepler's laws.** The following theorems were discovered by the astronomer Kepler after thirty years of study.

(1) The orbits of the planets are ellipses, the sun being in one focus.

(2) As a planet moves in its orbit, the radius vector from the sun describes equal areas in equal times.

(3) The squares of the periodic times of the several planets are proportional to the cubes of their major axes.

The last of these laws was published in 1619 in his *Harmonice Mundi* and the first two in 1609 in his work on *the motions of Mars.*

388. From the second of these laws, it follows that *the resultant force on each planet tends towards the sun;* Art. 307.

From the first we deduce that *the accelerating force on each planet is equal to* μ/r^2, where r is the instantaneous distance of that planet from the sun, and μ is a constant; Art. 332.

It is proved in Art. 341 that when the central force is μu^2, the periodic time in an ellipse is $T = 2\pi a^{\frac{3}{2}}/\sqrt{\mu}$, where a is the semi-major axis. Now Kepler's third law asserts that for all the planets T^2 is proportional to a^3; it follows that *μ is the same for all the planets.*

Laws corresponding to those of Kepler have been found to hold for the systems of planets and their satellites. Each satellite is therefore acted on by a force tending to the primary and that force follows the law of the inverse square.

It has been possible to trace out the paths of some of the comets and all these have been found to be conics having the sun in one focus. These bodies therefore move under the same law of force as the planets.

389. The laws of Kepler, being founded on observations, are not to be regarded as strictly true. They are approximations, whose errors, though small, are still perceptible. We learn from them that the sun, planets and satellites are so constituted that the sun may be regarded as attracting the planets, and the planets the satellites, according to the law of the inverse square. We now extend this law and make the *hypothesis* that the planets and satellites also attract the sun and attract each other according to the same law. Let us consider how this hypothesis may be tested.

Let m_1, m_2, &c. be certain constants, called the masses of the bodies, such that the accelerating attraction of the first on any other body distant r_1 is m_1/r_1^2, the attraction of the second is m_2/r_2^2, and so on. Let μ be the corresponding constant for the sun.

Assuming these accelerations, we can write down the differential equations of motion of the several bodies, regarded as particles. For example, the equations of motion of the particle m_1 may be obtained by equating d^2x/dt^2, &c., to the resolved accelerating attractions of the other bodies. The equations thus formed can only be solved by the method of continued approximation. Kepler's laws give us the first approximation; as a second approximation we take account of the attractions of the planets, but suppose that m_1, m_2, &c. are so small that the squares of their ratio to μ may be neglected. This problem is usually discussed in treatises on the Planetary theory. The solution of the problem enables us to calculate the positions of the planets and satellites at any given time and the results may be compared with their actual positions at that time. The comparison confirms the hypothesis in so extraordinary a way that we may consider its truth to be established as far as the solar system is concerned.

390. Extension to other systems. The law of gravitation being established for the solar system, its extension to other systems of stars may be only a fair inference. But we should notice that this extension is not founded on observation in the same sense that the truth of the law for the solar system is established*. The constituents of some double stars move round

* Villarceau, *Connaissance des temps* for the year 1852 published in 1849: A. Hall, *Gould's Astronomical Journal*, Boston, 1888.

each other in a periodic time sufficiently short to enable us to trace the changes in their distance and angular position. We may thus, partially at least, hope to verify the law of gravitation. What we see, however, is not the real path of either constituent, but its projection on the sphere of the heavens. We can determine if the relative path is a conic and can verify approximately the equable description of areas; but since the focus of the true path does not in general project into the focus of the visible path, an element of uncertainty as to the actual position of the centre of force is introduced.

We cannot therefore use Kepler's first law to deduce from these observations alone that the law of force is the inverse square.

391. Besides this, there are two practical difficulties. First, there is the delicacy of the observations, because the errors of observations bear a larger ratio to the quantities observed than in the solar system. Secondly, a considerable number of observations on each double star is necessary. Five conditions are required to fix the position of a conic, and the mean motion and epoch of the particle are also unknown. Unless therefore more than seven *distinct* observations have been made, we cannot verify that the path is a conic. These difficulties are gradually disappearing as observations accumulate and instruments are improved.

392. Besides the motions of the double stars we can only look to the proper motions of the stars in space for information on the law of gravitation. Some of these velocities are comparable to that of a comet in close proximity to the sun and yet there is no visible object in their neighbourhood to which we could ascribe the necessary attracting forces. At present no deductions can be made, we must wait till future observations have made clear the causes of the motions.

393. Other reasons. The law of gravitation is generally deduced from Kepler's laws, partly for historical reasons and partly because the proof is at once simple and complete. It is however useful and interesting to enquire what we may learn about the law of gravitation by considering other observed facts.

Ex. 1. It is given that for all initial conditions the path of a particle is a plane curve: deduce that the force is central.

Consider an orbit in a plane P, then at every point of that orbit the resultant force must lie in the plane. Taking any point A on the orbit project particles in all directions in that plane with arbitrary velocities, then since the plane of motion of each must contain the initial tangent at A and the direction of the force at A, each particle moves in the plane P. It follows that at every point of the plane P traversed by these orbits the resultant force lies in the plane. If these orbits do not cover the whole plane we take a new point B on the boundary of the area covered, and again project particles in all directions in that plane with arbitrary velocities. By continually repeating this process we can traverse every point of the plane, provided no points are separated from A by a line along which the

force is infinite. It follows that at every point of the plane P the force lies in that plane.

Next let us pass planes through any point A of one of these orbits and the direction AC of the force at A. Then by the same reasoning as before the direction of the force at points in each plane must lie in that plane and must therefore intersect AC. Thus the force at every point intersects the force at every other point. It follows that the force is central.

An observer placed at the sun, who noticed that all the planets described great circles in the heavens, would know from that one fact that the force acting on each was directed to the sun. Halphen, *Comptes Rendus*, vol. 84, Darboux's Notes to Despeyrous' *Mécanique*.

Ex. 2. If all the orbits in a given plane are conics, prove that the force is central.

If a particle P be projected from any point A in the direction of the force at A, the radius of curvature of the path is infinite at A. Since the only conic in which the radius of curvature is infinite is a straight line, the path of the particle P is a straight line and therefore the force at every point of this straight line acts along the straight line. The lines of force are therefore straight lines.

These straight lines could not have an envelope, for (unless the force at every point of that curve is infinite) we could project the particles along the tangents to the envelope past the point of contact so as to intersect other lines of force. The directions of the force would not then be the same at the same point for all paths. Bertrand, *Comptes Rendus*, vol. 84.

Ex. 3. If the orbits of all the double stars which have been observed are found to be closed curves, show that the Newtonian law of attraction may be extended to such bodies.

Bertrand has proved that all the orbits described about a centre of force (for all initial conditions within certain limits) cannot be closed unless the law of force is either the inverse square or the direct distance. By examining many cases of double stars we may include all varieties of initial conditions, and if all these orbits are closed the law of the inverse square may be rendered very probable. See Arts. 370, 426. Bertrand when giving this theorem in *Comptes Rendus*, vol. 77, 1873, quotes Tchebychef.

The Hodograph.

394. A straight line OQ is drawn from the origin O parallel to the instantaneous direction of motion and its length is proportional to the velocity of a particle P, say $OQ = kv$. The locus of Q has been called by Sir W. R. Hamilton the hodograph of the path of P. Its use is to exhibit to the eye the varying velocity and direction of motion of the particle. See Art. 29.

By giving k different values we have an infinite number of similar curves, any one of which may be used as a hodograph.

It follows from Art. 29 that, *if s' be the arc of the hodograph, ds'/dt represents in direction and magnitude the acceleration of P.*

395. If the force on the particle P is central and tends to the origin O, it is sometimes more convenient to draw OQ perpendicularly instead of parallel to the tangent. If OY be a perpendicular to the tangent, the velocity v of P is h/OY; hence if $OQ = kv$, we see that *the hodograph is then the polar reciprocal of the path with regard to the centre of force*, the radius of the auxiliary circle being $\sqrt{(hk)}$. If F be the central force at P, *the point Q travels along the hodograph with a velocity kF.*

396. Examples. *Ex.* 1. The path being an ellipse described about the centre C, and OQ being drawn parallel to the tangent, prove that the hodographs are similar ellipses.

Let CQ be the semi-conjugate of CP, then $v = \sqrt{\mu} \,.\, CQ$, Art. 326. Hence if $k = 1/\sqrt{\mu}$, *the hodograph is the ellipse itself.* The point Q then travels with a velocity $\sqrt{\mu} \,.\, CP$.

Ex. 2. The path being an ellipse described about the focus S, prove that a hodograph is the auxiliary circle, the other focus H being the origin and HQ drawn perpendicularly to the tangent at P.

Let SY, HZ be the two perpendiculars on the tangent, then $v = h/SY = HQ/k$, also $SY \,.\, HZ = b^2$, $\therefore HQ = HZ$ if $k = b^2/h$. Since the locus of Z is the auxiliary circle the result follows at once.

Ex. 3. The path being a parabola described about the near focus S, prove that a hodograph is the circle described on AS as diameter, where A is the vertex and SQ is drawn perpendicularly to the tangent.

Ex. 4. The hodograph of the path of a projectile is a vertical straight line, the radius vector OQ being drawn parallel to the tangent.

If the tangent at P make an angle ψ with the horizon, the abscissa of Q is $kv \cos \psi$. This is constant because the horizontal velocity of P is constant. The point Q travels along this straight line with a uniform velocity kg.

Ex. 5. An equiangular spiral is described about the pole, show that a hodograph is an equiangular spiral having the same pole and a supplementary angle. See Art. 30.

Ex. 6. A bead moves under the action of gravity along a smooth vertical circle starting from rest indefinitely near to the highest point. Show that a polar equation of a hodograph is $r' = b \sin \frac{1}{2}\theta'$, the origin being at the centre.

Ex. 7. The hodograph of the path of a particle P is given, show that if the path of P is a central orbit, the auxiliary point Q must travel along the hodograph with a velocity $v' = \lambda p'^2 \rho'$, where p' is the perpendicular from the centre of force on the tangent to the hodograph and ρ' is the radius of curvature. Show also that the central force $F = v'/k$ and the angular momentum $h = 1/\lambda k^3$.

The condition that the path is a central orbit is $v^2/\rho = Fp/r$. Writing $p = c^2/r'$ and $r = c^2/p'$, we find F and thence v'.

Ex. 8. The hodograph of the path of P is a parabola with its focus at O, and the radius vector $OQ=r'$ rotates with an angular velocity proportional to r'. Prove that the path of P is a circle passing through O, described about a centre of force situated at O.

Since the angular velocity of OQ is nr', we find by resolving v' perpendicularly to OQ that $v'=nr'^3/p'$. In a parabola $lr'=2p'^2$, and since $\rho'=r'dr'/dp'$ we see that $v'=\lambda p'^2\rho'$ where $\lambda=n/l$. The path is therefore a central orbit. But the polar reciprocal of $lr'=2p'^2$ (obtained by writing $p'=c^2/r$, and $r'=c^2/p$) is $r^2=p\,(2c^2/l)$, and this is a circle passing through O.

Ex. 9. A particle describes a curve under a constant acceleration which makes a constant angle with the tangent to the path; the motion takes place in a medium resisting as the nth power of the velocity. Show that the hodograph of the curve described is of the form $b^{-n}e^{-n\theta\cot\alpha}=r^{-n}-a^{-n}$. [Coll. Ex.]

Ex. 10. A particle, moving freely under the action of a force whose direction is always parallel to a fixed plane, describes a curve which lies on a right circular cone and crosses the generating lines at a constant angle. Prove that the hodograph is a conic section. [Coll. Ex.]

397. Elliptic velocity. Since the velocity is represented in direction and magnitude by the radius vector of the hodograph we may use the triangle of velocities to resolve the velocity into convenient directions.

Thus when the path is an ellipse described about the focus S, the velocity is represented perpendicularly by HZ/k, where $k=b^2/h$ and H is the other focus. If C be the centre this may be resolved into the constant lengths HC, CZ, the former being a part of the major axis and the latter being parallel to the radius vector SP. Hence *the velocity in an ellipse described about the focus S can be resolved into two constant velocities one equal to ae/k in a fixed direction, viz. perpendicular to the major axis, and the other equal to a/k in a direction perpendicular to the radius vector SP of the particle, where $k=b^2/h$.* [Frost's *Newton*, 1854.]

398. The hodograph an orbit. We have seen that when the force is central a hodograph of the path of P is a polar reciprocal. It follows that if the hodograph is the path of a second particle P', each curve is one hodograph of the other.

Ex. 1. Let r, r' be the radii vectores of any two corresponding points P, Q of a curve and its polar reciprocal, the radius of the auxiliary circle being c. If these curves be described by two particles P, P' with angular momenta h, h', prove that the central forces at the two points P, Q are connected by $FF'=\dfrac{h^2h'^2}{c^8}rr'$.

Ex. 2. Prove that the two particles will not continue to be at points which correspond geometrically in taking the polar reciprocal, unless the orbit of each is an ellipse described about the centre. [The necessary condition is that the velocity $v'=kF$ in the hodograph should be equal to the velocity $v'=h'/p'$ in the orbit. Since $p'=c^2/r$, this proves that F varies as r.]

Motion of two or more attracting Particles.

399. Motion of two attracting particles. This is the problem of finding the motion of the sun and a single planet which mutually attract each other. To include the case of two suns revolving round each other, as some double stars are seen to do, we shall make no restriction as to the relative masses of the two particles. The problem can be discussed in two ways according as we require the relative motion of the two particles or the motion of each in space.

Let M, m be the masses of the sun and the planet, r their instantaneous distance. The accelerating attraction of the sun on the planet is M/r^2, that of the planet on the sun m/r^2.

Initially the sun and the planet have definite velocities. Let us apply to each an initial velocity (in addition to its own) equal and opposite to that of the sun; let us also continually apply to each an acceleration equal and opposite to that produced in the sun by the planet's attraction. The sun will then be placed initially at rest, and will remain at rest, while *the relative motion of the planet will be unaltered.* See Art. 39.

The planet being now acted on by the two forces M/r^2 and m/r^2, both tending towards the sun, the whole force is $(M+m)/r^2$. The planet therefore, as seen from the sun, moves in an ellipse having the sun in one focus. The period is

$$\frac{2\pi}{\sqrt{(M+m)}}a^{\frac{3}{2}},$$

where a is the semi-major axis of the relative orbit. In the same way the sun, as seen from the planet, appears to describe an ellipse of the same size in the same time.

400. We notice that *the periodic time of a double star does not depend on the mass of either constituent, but on the sum of the masses.* The time in the same orbit is the same for the same total mass however that mass is distributed over the two bodies.

401. Consider next *the actual motion in space of the two particles.* We know by Art. 92 that the centre of gravity of the two bodies is either at rest or moves in a straight line with

uniform velocity. It is sufficient to investigate the motion relatively to the centre of gravity, for, when this is known, the actual motion may be constructed by imposing on each member of the system an additional velocity equal and parallel to that of the centre of gravity.

Let S and P be the sun and planet, G the centre of gravity, then $M \, . \, SP = (M+m) \, GP$. The attraction of the sun on the planet is

$$\frac{M}{SP^2} = \frac{M^3}{(M+m)^2} \frac{1}{GP^2} = \frac{M'}{GP^2}.$$

The attraction of the sun on the planet therefore tends to a point G fixed in space and follows the law of the inverse square. The planet therefore describes an ellipse in space with the centre of gravity in one focus, and the period is $\frac{2\pi}{\sqrt{M'}} \alpha^{\frac{3}{2}}$, where α is the semi-major axis of its actual orbit in space.

The actual orbits described by the sun and planet in space are obviously similar to each other and to the relative orbit of each about the other. If α, α' be the semi-major axes of the actual orbits of the planet and sun, a that of the relative orbit, we have by obvious properties of the centre of gravity,

$$\alpha/M = \alpha'/m = a/(M+m).$$

402. *To find the mass of a planet which has a satellite.* Since the mean accelerating attractions of the sun on the two bodies are nearly equal, their relative motion is also nearly the same as if the sun were away. Taking the relative orbit to be an ellipse, let a' be its semi-major axis. If m, m' are the masses of the planet and satellite, T' the period, we have $T'^2 = \frac{4\pi^2}{m+m'} a'^3$. When T' and a' have been found by observation, this formula gives the sum of the masses. The masses in this equation are measured in astronomical units, i.e. they are measured by the attractions of the bodies on a given supposititious particle placed at a given distance. It is therefore necessary to discover this unit by finding the attraction of some known body.

Consider the orbit described by the planet round the sun. Since we can neglect the disturbing attraction of the satellite,

we have, if a is the semi-major axis of the relative orbit and T the period, $T^2 = \frac{4\pi^2}{M+m} a^3$.

Dividing one of these equations by the other, we find

$$\frac{m+m'}{M+m} = \left(\frac{T}{T'}\right)^2 \left(\frac{a'}{a}\right)^3.$$

This formula contains only a ratio of masses, a ratio of times and a ratio of lengths. Whatever units these quantities are respectively measured in, the equation remains unaltered. Since m is small compared with the mass M of the sun, and m' small compared with the mass m of the primary, we may take as a near approximation $\frac{m}{M} = \left(\frac{T}{T'}\right)^2 \left(\frac{a'}{a}\right)^3$. In this way the ratio of the mass of any planet with a satellite to that of the sun can be found.

403. The determination of the mass of a planet without a satellite is very difficult, as it must be deduced from the perturbations of the neighbouring planets. Before the discovery of the satellites of Mars, Leverrier had been making the perturbations due to that planet his study for many years. It was only after a laborious and intricate calculation that he arrived at a determination of the mass. After Asaph Hall had discovered Deimos and Phobos the calculation could be shortly and effectively made. According to Asaph Hall the mass of Mars is 1/3,093,500 of the sun, while Leverrier made it about one three-millionth. This close agreement between two such different lines of investigation is very remarkable; see Art. 57. The minuteness of either satellite enables us to neglect the unknown ratio m'/m in Art. 402 and thus to determine the mass of Mars with great accuracy.

404. Examples. *Ex.* 1. Supposing the period of the earth round the sun and that of the moon round the earth to be roughly $365\frac{1}{4}$ and $27\frac{1}{3}$ days and the ratio of the mean distances to be 385, find the ratio of the sum of the masses of the earth and moon to that of the sun. The actual ratio given in the *Nautical Almanac* for 1899 is 1/328129.

Ex. 2. The constituents of a double star describe circles about each other in a time T. If they were deprived of velocity and allowed to drop into each other, prove that they will meet after a time $T/4\sqrt{2}$.

Ex. 3. The relative path of two mutually attracting particles is a circle of radius b. Prove that if the velocity of each is halved, the eccentricity of the subsequent relative path is 3/4 and the semi-major axis is $4b/7$.

Ex. 4. Two particles of masses m, m', which attract each other according to the Newtonian law, are describing relatively to each other elliptic orbits of major axis $2a$ and eccentricity e, and are at a distance r when one of them, viz. m, is suddenly fixed. Prove that the other will describe a conic of eccentricity e' such that

$$(m+m')\left\{\frac{2}{r}-\frac{(m+m')(1-e'^2)}{am(1-e^2)}\right\}=m\left(\frac{2}{r}-\frac{1}{a}\right).$$

It is supposed that the centre of gravity had no velocity at the instant before the particle m became fixed. [Coll. Ex. 1895.]

Ex. 5. Two particles move under the influence of gravity and of their mutual attractions: prove that their centre of gravity will describe a parabola and that each particle will describe relatively to that point areas proportional to the time. [Math. Tripos, 1860.]

Ex. 6. The coordinates of the simultaneous positions of two equal particles are given by the equations

$$x=a\theta-2a\sin\theta,\quad y=a-a\cos\theta;\qquad x_1=a\theta,\quad y_1=-a+a\cos\theta.$$

Prove that if they move under their mutual attractions, the law of force will be that of the inverse fifth power of the distance. [Math. Tripos.]

Ex. 7. Two homogeneous imperfectly elastic smooth spheres, which attract one another with a force in the line of their centres inversely proportional to the square of the distance between their centres, move under their mutual attraction, and a succession of oblique impacts takes place between them; prove that the tangents of the halves of the angles through which the line of centres turns between successive impacts diminish in geometrical progression. [Math. T. 1895.]

Consider the relative motion. The blow at each impact acts along the line joining the centres, hence the latera recta of all the ellipses described between successive impacts are equal. The normal relative velocity is multiplied by the coefficient of elasticity at each impact. The radius vector of the relative ellipse is the same at each impact, being the sum of the radii of the spheres. The result follows immediately from Ex. 1, Art. 337.

405. *Ex.* 1. Herschel says that the star Algol is usually visible as a star of the second magnitude and continues such for the space of 2 days $13\frac{1}{2}$ hours. It then suddenly begins to diminish in splendour and in $3\frac{1}{2}$ hours is reduced to the fourth magnitude, at which it continues for about 15 minutes. It then begins to increase again and in $3\frac{1}{2}$ hours more is restored to its usual brightness, going through all its changes in 2 d. 20 hr. 48 min. 54·7 sec. This is supposed to be due to the revolution round it of some opaque body which, when interposed between us and Algol, cuts off a portion of the light. Supposing the brilliancy of a star of the second magnitude to be to that of the fourth as 40 to 6·3 and that the relative orbit of the bodies is nearly circular and has the earth in its plane, prove that the radii of the two constituents of Algol are as 100 : 92 and that the ratios of their radii to that of their relative orbit are equal to ·171 and ·160. If the radius of the sun be 430000 miles and its density be 1·444, taking water as the unit, prove that the density of either constituent of Algol (taking them to be of equal densities) is one-fourth that of water. The numbers are only approximate.

[Maxwell Hall, *Observatory*, 1886.]

Ex. 2. The brightness of a variable star undergoes a periodic series of changes in a period of T years. The brightness remains constant for mT years, then gradually diminishes to a minimum value, equal to $1-k^2$ of the maximum, at which minimum it remains constant for nT years and then gradually rises to the original maximum. Show that these changes can be explained on the hypothesis that a dark satellite revolves round the star. Prove also that, if the relative orbit is circular, and the two stars are spherical, the ratio of the mean density of the double star to that of the sun is

$$\frac{\sin^3 \frac{1}{2} D}{T^2 (1+k^3)} \left[\frac{(1+k)^2 \cos^2 n\pi - (1-k)^2 \cos^2 m\pi}{\cos^2 n\pi - \cos^2 m\pi}\right]^{\frac{3}{2}},$$

where D is the apparent diameter of the sun at its mean distance. [Math. T. 1893.]

406. Three attracting Particles. The problem of determining the relative motions of three or more attracting particles has not been generally solved. The various solutions in series which have as yet been obtained usually form the subjects of separate treatises, and are called the Lunar and Planetary theories. Laplace has however shown that there are some cases in which the problem can be accurately solved in finite terms*.

407. Let the several particles be so arranged in a plane that the resultant accelerating force on each passes through the common centre of gravity O of the system and that each resultant is proportional to the distance of the particle from that centre. It is then evident that if the proper common angular velocity be given to the system about O, the centrifugal force on each particle may be made to balance the attraction on that particle. The particles of the system will then move in circles round O with equal angular velocities, the lines joining them forming a figure always equal and similar to itself. Each particle also will describe a circle relatively to any other particle.

Let us next enquire what conditions are necessary that the particles may so move that the figure formed by them is always similar to its original shape, but *of varying size*. Let the distances

* Laplace's discussion may be found in the sixth chapter of the tenth book of the *Mécanique Céleste*. The proposition that the motion when the particles are in a straight line is unstable was first established by Liouville, *Académie des Sciences*, 1842, and *Connaissance des Temps* for 1845 published in 1842. His proof is different from that given in the text. The motion when the particles are at the corners of an equilateral triangle is discussed in the *Proceedings of the London Mathematical Society*, Feb. 1875. See also the author's *Rigid Dynamics*, vol. I. Art. 286, and vol. II. Art. 108. There is also a paper by A. G. Wythoff, *On the Dynamical stability of a system of particles*, *Amsterdam Math. Soc.* 1896.

of the particles from the centre of gravity O be r_1, r_2, &c. We then have for each particle the equations

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -F, \quad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = G.$$

Since the figure is always similar, these equations are to be satisfied when $d\theta/dt$ is the same for every particle, and r_1, r_2, &c. have the ratios α_1, α_2, &c., where α_1, α_2, &c., are some positive finite constant quantities. It immediately follows that the arrangement must be such that the F's are in the same positive ratios and also the G's.

Since the mutual attractions of the particles form a system of forces in equilibrium, the equivalent system m_1F_1, m_2F_2, &c. and m_1G_1, m_2G_2, &c. is also in equilibrium. The sum of the moments of the G's about O must therefore be zero, which (since they are in the ratios α_1, &c.) is impossible unless each G is zero.

If also the initial conditions are such that both the radial velocities dr_1/dt, &c. and the transverse velocities $r_1d\theta/dt$, &c., have the ratios α_1, &c., all the equations will be satisfied by assuming r_1, r_2, &c. to have the constant ratios α_1, α_2, &c. The motion of some one particle, say m_1, is determined by the two polar equations of that particle

The result is, that if the particles move so as to be always at the corners of a similar figure, that figure must be such that the resultant accelerating forces on the particles act towards the common centre of gravity O and are proportional to the distances from O. This being true initially, the particles must be projected in directions making equal angles in the same sense with their distances from O, with velocities proportional to those distances.

408. The two arrangements. *To determine how three particles must be arranged so that the force on any one may pass through the common centre of gravity; the law of force being the inverse κth power of the distance.*

It is evident that the condition is satisfied when the three particles are arranged in a straight line. We have now to enquire if any other arrangement is possible.

It is a known theorem in attraction that if two given particles of masses M, m attract a third m', placed at distances ρ, r from them, with accelerating forces $M\rho$, mr, the resultant passes through

the centre of gravity of M, m and therefore through that of all three. In order that the resultant of M/ρ^κ and m/r^κ may also pass through the centre of gravity of M, m, it is evident that the ratio of M/ρ^κ to m/r^κ must be equal to the ratio $M\rho$ to mr. It immediately follows (except $\kappa = -1$) that $\rho = r$. The three particles must therefore be at equal distances; see also Art. 304.

The result is that for three attracting particles there are only two possible arrangements; (1) that in which the particles, however unequal their masses may be, are at the corners of an equilateral triangle, (2) that in which they are in the same straight line.

It may also be shown that when the law of attraction is the inverse κth, *the arrangement at the corners of an equilateral triangle is stable when* $\dfrac{(\Sigma m)^2}{\Sigma mm'} > 3\left(\dfrac{1+\kappa}{3-\kappa}\right)^2$.

409. The line arrangement. *Three mutually attracting particles whose masses are M, m', m are placed in a straight line. It is required to determine the conditions that throughout their subsequent motion they may remain in a straight line.*

Let the law of attraction be the inverse κth power of the distance. Let M, m, be the two extreme particles, m' being between the other two. Let a, b, c be the distances Mm, Mm', $m'm$; then $a = b + c$.

A necessary condition is that the resultant accelerating forces on the particles must be proportional to their distances from the centre of gravity O (Art. 407). We therefore have

$$\frac{M/a^\kappa + m'/c^\kappa}{Ma + m'c} = \frac{M/b^\kappa - m/c^\kappa}{Mb - mc} = \frac{m/a^\kappa + m'/b^\kappa}{ma + m'b} \quad \text{.........}(1),$$

where the numerators express the accelerating forces on the particles and the denominators are proportional to the distances from O.

The equalities (1) are equivalent to only one equation, for if we multiply the numerators and denominators of the three fractions by m, m', $-M$ respectively, the sum of the numerators and also that of the denominators are zero. Putting $a = b(1+p)$, $c = bp$, we arrive at

$$Mp^\kappa\{(1+p)^{\kappa+1} - 1\} - m'(1+p)^\kappa(1-p^{\kappa+1}) - m\{(1+p)^{\kappa+1} - p^{\kappa+1}\} = 0 \ldots(2).$$

The left-hand side is negative when $p = 0$ and positive when p is

infinitely large, *the equation therefore has one real positive root, whatever positive values M, m', m may have.* Putting $p=1$, the left side becomes $(M-m)(2^{\kappa+1}-1)$; since we may take M as the greater of the two extreme particles we see that *the real positive value of p is less than unity, provided $\kappa+1$ is positive.* If $\kappa+1$ were negative the root would be greater than unity.

Whatever the masses of the particles may be it follows that if they are so placed that their distances have the ratios given by this value of p, and their parallel velocities are proportional to their distances from O, they will throughout their subsequent motion remain in a straight line.

When the attraction follows the Newtonian law, the equation (2) becomes the quintic

$$(M+m')p^5+(3M+2m')p^4+(3M+m')p^3-(m'+3m)p^2$$
$$-(2m'+3m)p-(m+m')=0\ldots(3).$$

The terms of this equation exhibit but one variation of sign, and there is therefore but one positive root.

It may be shown in exactly the same way that in the general case, when κ has any positive integral value, the equation (2) has only one positive root; all the terms from $p^{2\kappa+1}$ to $p^{\kappa+1}$ being positive, while those from p^{κ} to p^0 are negative.

410. When the positions of two of the masses are given, there are three possible cases; according as the third is between the other two or on either side. Since the analytical expression for the law of the inverse square does not represent the attraction when the attracted particle passes through the centre of force, Art. 135; these three cases cannot be included in the same equation. We thus have three equations of the form (3), one for each arrangement.

411. In the case of the sun, earth, and moon, M is very much greater than either m or m'. Since p vanishes when m and m' are zero, we infer that p is very small when m/M and m'/M are small. The equation (3) therefore gives $3p^3=(m+m')/M$, or, using the numerical values of m, m' and M, $p=1/100$ nearly.

If the moon were therefore placed at a distance from the earth one hundredth part of that of the sun, the three bodies might be projected so that they would always remain in a straight

line. The moon would then be always full, but at that distance its light would be much diminished. This configuration of the sun, earth and moon however could not occur in nature because this state of steady motion is unstable. On the slightest disturbance the whole system would change and the particles would widely deviate from their former paths.

412. *Three mutually attracting particles whose masses are M, m', m describe circles round their common centre of gravity and are always in a straight line. Prove that if the force vary as any inverse power of the distance this state of motion is unstable.*

Reducing the particle M to rest we take that point as the origin of coordinates. Let (r, θ) be the coordinates of m, (r', θ') those of m'. The particle m is acted on by $(M+m)/r^\kappa$ along the straight line mM, and m'/r'^κ in a direction parallel to $m'M$. The polar equations of the motion of m are

$$\left.\begin{aligned} \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 &= -\frac{M+m}{r^\kappa} - \frac{m'}{r'^\kappa}\cos\omega - \frac{m'}{R^\kappa}\cos\phi \\ \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) &= -\frac{m'}{r'^\kappa}\sin\omega + \frac{m'}{R^\kappa}\frac{r'\sin\omega}{R} \end{aligned}\right\} \qquad (1),$$

where ω, ϕ are the angles at M, m of the triangle formed by joining the particles and R is the side mm'. In the same way the polar equations of the motion of m' are

$$\left.\begin{aligned} \frac{d^2r'}{dt^2} - r'\left(\frac{d\theta'}{dt}\right)^2 &= -\frac{M+m'}{r'^\kappa} - \frac{m}{r^\kappa}\cos\omega + \frac{m}{R^\kappa}\cos\phi' \\ \frac{1}{r'}\frac{d}{dt}\left(r'^2\frac{d\theta'}{dt}\right) &= \frac{m}{r^\kappa}\sin\omega - \frac{m}{R^\kappa}\sin\phi' \end{aligned}\right\} \qquad (2),$$

where ϕ' is the external angle of the triangle at m'. In forming these equations the standard case is that in which $\theta' > \theta$ and $r' < r$.

We shall now substitute in these equations $r = a(1+x)$, $\theta = nt + y$; $r' = b(1+y)$, $\theta' = nt + \eta$, and reject all powers beyond the first of the small quantities x, y, ξ, η. Remembering that $\sin\phi/r' = \sin\phi'/r = \sin\omega/R$ we find after some reduction

$$\begin{aligned} (\delta^2 - n^2 - \kappa E)\,x - 2n\delta y + m'\kappa B\xi + 0\,.\,\eta &= 0, \\ 2n\delta x + (\delta^2 + m'B)\,y + 0\,.\,\xi - m'B\eta &= 0, \\ m\kappa Ax + 0\,.\,y + (\delta^2 - n^2 - \kappa F)\,\xi - 2n\delta\eta &= 0, \\ 0\,.\,x - mAy + 2n\delta\xi + (\delta^2 + mA)\,\eta &= 0, \end{aligned}$$

where for brevity we have written δ for d/dt, and $c = a - b$,

$$A = \frac{a}{b}\left(\frac{1}{c^{\kappa+1}} - \frac{1}{a^{\kappa+1}}\right), \qquad B = \frac{b}{a}\left(\frac{1}{c^{\kappa+1}} - \frac{1}{b^{\kappa+1}}\right),$$

$$E = \frac{M+m}{a^{\kappa+1}} + \frac{m'}{c^{\kappa+1}}, \qquad F = \frac{M+m'}{b^{\kappa+1}} + \frac{m}{c^{\kappa+1}}.$$

The steady motion has been already found in Art. 409, but it may also be deduced from the first and third of the equations (1) and (2) by equating the constants. We thus find $n^2 = E - m'B$, $n^2 = F - mA$.

We notice that the constants E, F are positive. When $\kappa+1$ is positive, it has been shown in Art. 409 that $a>b>c$, and therefore A, B and $E+F-2n^2$ are positive. Lastly whatever κ may be $E+F-n^2$ is positive

To solve the four equations, we put $x=Ge^{\lambda t}$, $y=He^{\lambda t}$, $\xi=Ke^{\lambda t}$, $\eta=Le^{\lambda t}$. Substituting and eliminating the ratios G, H, K, L we obtain a determinantal equation whose constituents are the coefficients of x, y, ξ, η, with λ written for δ. This determinant is of the eighth degree in λ. To find its factors we must before expansion make some necessary simplifications which we can only indicate here. We first add the ξ column to the x column and the η column to the y column. The second column may now be divided by λ. Multiplying the second column by $2n$ and subtracting from the first, we see that $\lambda^2-(\kappa-3)\,n^2$ is another factor which we divide out. Subtracting the first row from the third and the second from the fourth, the first column acquires three zeros and the second column two. The determinant is now easily expanded and we have

$$\lambda^2\,\{\lambda^2-(\kappa-3)\,n^2\}\,\{(\lambda^2+C)\,(\lambda^2-C\kappa-(\kappa+1)\,n^2)+4n^2\lambda^2\}=0,$$

where $C=E+F-2n^2$. If $\kappa>3$, this equation gives a real positive value of λ and the motion is therefore unstable. If κ have any positive value C is positive, and the third factor has the product of its roots negative; one value of λ^2 is real and positive and the other real and negative. *The motion is therefore unstable for all positive values of κ.*

413. *Ex.* 1. Three mutually attracting particles are placed at rest in a straight line. Show that they will simultaneously impinge on each other if the initial distances apart are given by the value of p in the equation of the $(2\kappa+1)$th degree of Art. 409. [This equation expresses the condition that the distances between the particles are always in a constant ratio.]

Ex. 2. Three unequal mutually attracting particles are placed at rest at the corners of an equilateral triangle and attract each other according to the inverse κth power of the distances. Prove that they will arrive simultaneously at the common centre of gravity. If the law of attraction is the inverse square, the time of transit is $\frac{1}{2}\pi\,(a^3/2\mu)^{\frac{1}{2}}$ where μ is the sum of the masses and a the side of the initial triangle, Art. 131.

414. A swarm of particles. Let us suppose that a comet is an aggregation of particles whose centre of gravity describes an elliptic orbit round the sun. The question arises, what are the conditions that such a swarm could keep together*? Similar conditions must be satisfied in the case of a swarm consolidating

* The disintegration of comets was first suggested by Schiaparelli who proved that the disturbing force of the sun on a particle might be greater than the attraction of the comet. He thus obtained as a necessary condition of stability $m/b^3>2M/a^3$. The subject was dynamically treated by Charlier and Luc Picart on the supposition of a circular trajectory. They arrived at the condition $m/b^3>3M/a^3$; *Bulletin de l'Académie de S. Pétersbourg*, *Annales de l'Observatoire de Bordeaux*, Tisserand, *Méc. Céleste*, IV. The condition of stability was extended to the case of an elliptic trajectory by M. O. Callandreau in the *Bulletin Astronomique*, 1896. The brief solutions here given of these problems are simplifications of their methods.

into a planet in obedience to the Nebular theory. The following example will illustrate the method of proceeding.

We shall suppose the sun A to be fixed in space, Art. 399. Let B be the centre of the swarm, C any particle. Let r, θ be the polar coordinates of B referred to A, and ξ, η the coordinates of C referred to B as origin, the axis of ξ being the prolongation of AB. Let M be the mass of the sun. Supposing, as a first approximation, that the swarm is homogeneous and spherical, its attraction at an *internal point* C is $\mu\rho$, where $\rho = BC$. If m be the mass and b the radius of the swarm, $\mu b = m/b^2$.

The equations of motion are, by Art. 227,

$$\left.\begin{aligned}\frac{d^2(r+\xi)}{dt^2} - (r+\xi)\left(\frac{d\theta}{dt}\right)^2 - \frac{1}{\eta}\frac{d}{dt}\left(\eta^2\frac{d\theta}{dt}\right) &= \frac{-M}{(r+\xi)^2} - \mu\xi \\ \frac{d^2\eta}{dt^2} - \eta\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{r+\xi}\frac{d}{dt}\left\{(r+\xi)^2\frac{d\theta}{dt}\right\} &= \frac{-M\eta}{(r+\xi)^3} - \mu\eta\end{aligned}\right\} \quad \text{...(1)}.$$

These equations also apply to the motion of the particle at B, where $\xi = 0$, $\eta = 0$. Hence when we expand in powers of ξ, η, all the terms independent of ξ, η must cancel out. We thus have

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2} - 2\frac{d\eta}{dt}\frac{d\theta}{dt} - \eta\frac{d^2\theta}{dt^2} - \xi\left(\frac{d\theta}{dt}\right)^2 &= \frac{2M\xi}{r^3} - \mu\xi \\ \frac{d^2\eta}{dt^2} + 2\frac{d\xi}{dt}\frac{d\theta}{dt} + \xi\frac{d^2\theta}{dt^2} - \eta\left(\frac{d\theta}{dt}\right)^2 &= \frac{-M\eta}{r^3} - \mu\eta\end{aligned}\right\} \quad \text{......(2)}.$$

If the centre of gravity of the swarm describe a circle about the sun, we write $r = a$, $d\theta/dt = n$. The equations then become

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2} - 2n\frac{d\eta}{dt} + (\mu - 3n^2)\,\xi &= 0 \\ \frac{d^2\eta}{dt^2} + 2n\frac{d\xi}{dt} + \mu\eta \quad &= 0\end{aligned}\right\} \quad \text{..............(3)}$$

Putting $\xi = A\cos(pt+\alpha)$, $\eta = B\sin(pt+\alpha)$, we immediately obtain the determinantal equation

$$(p^2 - \mu + 3n^2)(p^2 - \mu) - 4p^2n^2 = 0 \text{..............(4)}.$$

The condition that the particles of the swarm should keep together is the same as the condition that the roots of this quadratic should be real and positive. The left-hand side is positive when $p^2 = \pm\infty$, and negative when $p^2 = \mu$ and $p^2 = \mu - 3n^2$. The required condition

is therefore $\mu > 3n^2$, Art. 288. *The condition that the swarm is stable is therefore* $\frac{m}{b^3} > 3\frac{M}{a^3}$.

Unless therefore the density of the swarm exceed a certain quantity the swarm cannot be stable. If the mass of the sun were distributed throughout the sphere whose radius is such that the swarm is on the surface, the density of the swarm must be at least three times that of the sphere.

The path of the particle C when describing either principal oscillation is (relatively to the axes $B\xi$, $B\eta$) an ellipse with its centre at B. Substituting the values of ξ, η in the equations of motion and using the quadratic, we find

$$\frac{A}{B}=\frac{p^2-\mu}{-2np}, \quad \frac{A^2}{B^2}=1-\frac{3}{4}\frac{p^2-\mu}{p^2}, \quad \frac{A_1A_2}{B_1B_2}=-\sqrt{\frac{\mu}{\mu-3n^2}}.$$

Since μ lies between the values of p^2, the first equation shows that A_1/B_1 and A_2/B_2 have opposite signs, and accordingly the radical is negative.

It follows that the oscillation which corresponds to the smaller value of p has the major axis directed along $B\xi$, while in the other that axis is along $B\eta$. The particle also describes the ellipses in opposite directions, in the former case the direction is the same as that of the swarm round the sun, in the latter, the opposite.

If the centre of gravity of the swarm describe an ellipse of small eccentricity, we may obtain an approximate solution of the equations of motion. Assuming the expansions $\theta = nt + 2e\sin nt + \frac{5}{4}e^2\sin 2nt$,

$$\frac{h}{r^2}=\frac{d\theta}{dt}; \quad \therefore \left(\frac{a}{r}\right)^3 = 1+3e\cos nt+\tfrac{3}{2}e^2+\tfrac{9}{2}e^2\cos 2nt,$$

it is evident that all the coefficients of the differential equations (2) can be at once expressed in terms of t, including all terms which contain e^2. It is however unnecessary for our present purpose to write these at length. It is easy to see that the equations become

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2}-2n\frac{d\eta}{dt}+\{\mu-n^2(3+5e^2)\}\,\xi &= eX\\ \frac{d^2\eta}{dt^2}+2n\frac{d\xi}{dt}+\{\mu-\tfrac{1}{2}n^2e^2\}\,\eta &= eY\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots (5),$$

$$eX = 4en\cos nt\,\frac{d\eta}{dt} - 2en^2\sin nt\,\eta + 10en^2\cos nt\,\xi + \&c.,$$

$$eY = -4en\cos nt\,\frac{d\xi}{dt} + 2en^2\sin nt\,\xi + en^2\cos nt\,\eta + \&c.$$

As a first approximation we neglect eX, eY. Comparing the equations (5) and (3) we see at once that we shall have the quadratic

$$\{p^2-\mu+n^2(3+5e^2)\}\,\{p^2-\mu+\tfrac{1}{2}n^2e^2\}-4p^2n^2=0 \ldots\ldots\ldots\ldots (6).$$

The condition that the swarm is stable is then $\mu > n^2(3+5e^2)$; $\therefore \frac{m}{b^3} > \frac{M}{a^3}(3+5e^2)$.

It appears therefore that *the gradual dissipation of a comet is more probable when the trajectory is elliptical than when it is circular.*

As a second approximation, we substitute $\xi = A\cos(pt+\alpha)$, $\eta = B\sin(pt+\alpha)$ in the expressions X and Y. By Art. 303 the only important terms are those which become magnified by the process of solution. These terms are of the form $P\cos(\lambda t+L)$ where $\lambda = p\pm n$ or $p\pm 2n$. Unless therefore the roots p, p' of the quadratic (6) or (4) are such that $p\pm p'$ is nearly equal to n or $2n$, the terms derived from X, Y remain respectively of the order e or e^2. This relation between the roots cannot occur when e is small.

415. Tisserand's criterion*. When a comet describing a conic round the sun passes very near to a planet, such as Jupiter, its course is much disturbed. When it emerges from the sphere of perceptible influence of the planet, it may again be supposed to describe a conic round the sun, but the elements of the new path may be very different from those of the old.

Since Jacobi's integral (Art. 255) holds throughout the motion, the elements of both the conics must satisfy that equation.

Let (a_0, l_0), (a_1, l_1) be the semi-major axis and semi-latus rectum before and after passing through the sphere of influence of the planet. Let i_0, i_1 be the inclinations of the planes of the comet's orbit to the plane of the planet's motion.

Let the sun O be taken as the origin of coordinates, and let the axis of ξ pass through the planet P. Let r, ρ be the distances of the comet Q from O and P respectively and $c = OP$. Let M, m be the masses of the sun and planet, then, reducing the sun to rest (Art. 399), we regard the comet as acted on by the resultant attraction of the sun and planet together with a force m/c^2 acting parallel to PO. The field of force is therefore defined by

$$U = \frac{M}{r} + \frac{m}{\rho} - \frac{m\xi}{c^2}.$$

We suppose that the planet P describes a circular orbit relatively to O with a constant angular velocity n, where $n^2 = (M+m)/c^3$. The Jacobian integral takes the form

$$\tfrac{1}{2}V^2 - nA - \frac{M}{r} - \frac{m}{\rho} + \frac{m\xi}{c^2} = C,$$

* Tisserand's criterion may be found in his Note sur l'integrale de Jacobi, et sur son application à la théorie des comètes, *Bulletin Astronomique*, Tome VI. 1889, also in his *Mécanique Céleste*, Tome IV. 1896. M. O. Callandreau's addition is given in the second chapter of his Étude sur la théorie des comètes périodiques, *Annales de l'Observatoire de Paris, Mémoires*, 1892, Tome XX. There are also some investigations by H. A. Newton on the capture of comets by planets, especially Jupiter, *American Journal of Science*, vol. XLII. pages 183 and 482, 1891.

where V is the space velocity of the comet and A its angular momentum referred to a unit of mass. Since (Art. 333)

$$V^2 = M\left(\frac{2}{r} - \frac{1}{a}\right), \quad A = \cos i \sqrt{(Ml)},$$

the integral becomes

$$\frac{1}{2a_0} + n\cos i_0 \sqrt{\frac{l_0}{M}} + \frac{m}{M}\left(\frac{1}{\rho_0} - \frac{\xi_0}{c^2}\right) = \frac{1}{2a_1} + n\cos i_1 \sqrt{\frac{l_1}{M}} + \frac{m}{M}\left(\frac{1}{\rho_1} - \frac{\xi_1}{c^2}\right),$$

where ξ_0, ρ_0; ξ_1, ρ_1, are the values of ξ, ρ when the comet is respectively entering and leaving the sphere of influence of the planet. We obviously have $\rho_0 = \rho_1$, and since the comet does not stay long within the sphere, we may neglect $\xi_0 - \xi_1$ when multiplied by the very small quantity m/M. Writing then $n^2 = M/c^3$ as a close approximation, Art. 341, we obtain the criterion

$$\frac{1}{2a_0} + \frac{\cos i_0 \sqrt{l_0}}{c\sqrt{c}} = \frac{1}{2a_1} + \frac{\cos i_1 \sqrt{l_1}}{c\sqrt{c}}.$$

416. Tisserand uses this criterion to determine whether two comets both of which are known to have passed near Jupiter could be the same body. If the criterion is not satisfied by the known elements of the two comets, they cannot be the same body. If it is satisfied it is then worth while to examine more thoroughly how much the elements of either body have been altered by the attraction of Jupiter. This must be done by using the method of the planetary theory and is generally a laborious process.

In Tisserand's criterion the orbit of Jupiter is considered to be circular, which is not strictly correct. This defect has been corrected by M. O. Callandreau. Taking account only of the first power of the eccentricity he adds a small term containing that eccentricity as a factor. This term, unlike those in Tisserand's criterion, depends on the manner in which the comet approaches Jupiter.

417. Stability deduced from Vis Viva. The Jacobian integral has been used by G. W. Hill* to determine whether the moon could be indefinitely pulled away from the earth by the disturbing attraction of the sun. In such a problem as this, it is convenient to take the origin at the earth P and the moving axis of ξ directed towards the sun O. Reducing the earth to rest, the moon Q is acted on by $(m+m')/\rho^2$ along QP and M/c^2 parallel to OP. The Jacobian equation for relative motion, Art. 255 (3), takes the form

$$\tfrac{1}{2}v^2 = \tfrac{1}{2}n^2\rho^2 + \frac{\mu}{\rho} + \frac{M}{r} - \frac{M}{c^2}\xi + C,$$

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* G. W. Hill's researches in the Lunar theory may be found in the *American Journal of Mathematics*, vol. I. 1878.

where $\rho = PQ$, $r = OQ$, $c = OP$ and μ is the sum of the masses m, m' of the earth and moon. We treat the sun's orbit as circular and put as a near approximation $M/c^3 = n^2$. Since $\rho^2 = \xi^2 + \eta^2$, this equation becomes

$$\tfrac{1}{2}v^2 = \frac{\mu}{\rho} + \frac{c^3 n^2}{r} + \frac{n^2}{2}\{(c-\xi)^2 + \eta^2\} - C'.$$

Since the left-hand side is essentially positive it is clear that *the moving particle Q can never cross the surface defined by equating the right-hand side to zero, and can only move in those parts of space in which the right-hand side is positive.* Art. 299.

If the initial circumstances of the motion make C' negative, the right-hand side is always positive and the equation supplies no limits to the position of Q.

The form of the surface when C' is positive has been discussed by Hill. When C' exceeds a certain quantity the surface has in general three separate sheets. The inner of these is smaller than the other two and surrounds the earth. The second is also closed but surrounds the sun, the third is not closed. When the constants are adapted to the case of the moon, that satellite is found to be within the first sheet. It must therefore always remain there, and its distance from the earth can never exceed 110 equatorial radii. Thus *the eccentricity of the earth's orbit being neglected, we have a rigorous demonstration of a superior limit to the radius vector of the moon.*

418. *Ex.* 1. If the moon Q move in the plane of motion of the earth P and if also the sun is so remote that we may put $\frac{c^3}{r} + \frac{1}{2}r^2 = \frac{3}{2}c^2\left(1 + \frac{\xi^2}{c^2}\right)$ when the left-hand side is expanded in powers of ξ/c and η/c, the bounding surface degenerates into the curve $\frac{\mu}{\rho} + \frac{3}{2}n^2\xi^2 = C''$. It is required to trace the forms of this curve for different positive values of C''.

The curve has two infinite branches tending to the asymptotes $\frac{3}{2}n^2\xi^2 = C''$. If C'' is greater than the minimum value of $\mu/\xi + \frac{3}{2}n^2\xi^2$ there is also an oval round the body S. If the particle Q is within the oval, it cannot escape thence and its radius vector will have a superior limit. If the particle is beyond either of the infinite branches, it cannot cross them and the radius vector will have an inferior limit. The velocity at any point of the space between the oval and the infinite branches is imaginary. [Hill.]

Ex. 2. A double star is formed by two equal constituents S, P whose orbits are circles. A third particle Q whose mass is infinitely small moves in the same plane and initially is at a distance from P on SP produced equal to half SP, starting with such velocity that it would have described a circular orbit about P if S had been absent. Show that the curve of no relative velocity is closed, and that the particle being initially within that curve cannot recede indefinitely from the attracting bodies S and P.

This example is discussed by Coculesco in the *Comptes Rendus*, 1892. He also refers to a memoir of M. de Haerdtl, 1890, where the revolution of Q round P is traced during two revolutions and it is shown that at the end of the third the particle is receding from A *.

* Since writing the above the author has received Darwin's memoir on Periodic Orbits, *Acta Mathematica*, XXI. in which the motion of a planet about a binary star

Theory of Apses.

419. *When the law of force is a one-valued function of the distance, every apsidal radius vector must divide the orbit symmetrically.*

Let O be the centre of force, A an apse (Art. 314). The argument rests on two propositions.

(1) If two particles are projected from A with equal velocities, both perpendicularly to OA but in opposite directions, it is clear that (the force being always the same at the same distance from O) the paths described must be symmetrical about OA.

(2) If at any point of its path, the velocity of the particle were reversed in direction (without changing its magnitude), the particle would describe the same path but in a reverse direction.

If then a particle describing an orbit arrive at an apse A, its subsequent path when reversed must be the same as its previous path. Hence OA divides the whole orbit symmetrically.

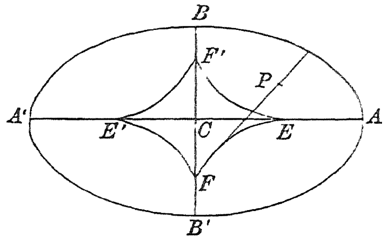
We may notice that if the law of force were not one-valued, say $F=\mu\{u \pm \sqrt{(u^2-a^2)}\}$, where the apsidal distance $OA=a$, the first proposition is not true, unless it is also given that the radical keeps one sign.

420. *There can be only two apsidal distances though there may be any number of apses.*

Let the particle after passing an apse A arrive at another apse B. Then since OB divides the orbit symmetrically, there must be a third apse C beyond B such that the angles AOB, BOC are equal and $OC=OA$. Since OC divides the orbit symmetrically, there is a fourth apse at D, where $OD=OB$ and the angles BOC, COD are equal. The apsidal distances are therefore alternately equal, and the angle contained at O by any two consecutive apsidal distances is always the same.

has been more thoroughly studied. Taking a variety of initial conditions he has traced the subsequent paths of a particle of insignificant mass. Some of the paths thus presented to the eye have such unexpected and remarkable forms that the paper is full of interest.

421. Examples. *Ex.* 1. Show that an ellipse cannot be described about a centre of force whose attraction is a one-valued function of the distance unless that centre is situated on a principal diameter and is outside the evolute.



By drawing all the tangents to one arc EF of the evolute we see that they cover the whole area of the quadrant ACB of the ellipse. It follows that a normal to the ellipse can be drawn through any point P situated in this quadrant, and this normal does not divide the ellipse symmetrically, unless P lies between E and A or between F' and B.

Ex. 2. If the path is an equiangular spiral and the central force a one-valued function of the distance, prove that the centre of force must be situated in the pole.

Ex. 3. If a particle of mass m be attached to a fine elastic string of natural length a and modulus λ, and lie with the string unstretched and one extremity fixed on a smooth horizontal plane; prove that, if projected at right angles to the string with velocity v, the string will just be doubled in length at its greatest extension if $3mv^2 = 4a\lambda$. [Coll. Ex.]

Ex. 4. A particle is projected from an apse with a velocity v, prove that the apse will be an apocentre or a pericentre according as the velocity v is less or greater than that in a circle at the same distance.

422. The apsidal distances. *To find the apsidal distances when $F = \mu u^n$, and n is an integer.*

The equation of vis viva, viz. $v^2 = C - 2\int F dr$, gives

$$v^2 = h^2\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} = C + \frac{2\mu}{n-1}u^{n-1} \quad\ldots\ldots\ldots\ldots(1).$$

Let V be the velocity at the initial distance R, β the angle of projection, then

$$V^2 = C + \frac{2\mu}{n-1}\left(\frac{1}{R}\right)^{n-1}, \qquad h = VR\sin\beta \quad\ldots\ldots\ldots(2).$$

Thus both h and C are known quantities, at an apse u is a max-min, and therefore $du/d\theta = 0$. The apsidal distances are therefore given by

$$\left(\frac{dr}{dt}\right)^2 = h^2\left(\frac{du}{d\theta}\right)^2 = \frac{2\mu}{n-1}u^{n-1} - h^2u^2 + C = 0 \quad\ldots\ldots\ (A).$$

If an equation is arranged in descending powers of the unknown quantity, we know by Descartes' theorem that there cannot be more positive roots than variations of sign. The arrangement of the terms of equation (A) will depend on whether $n-1$ is greater

or less than 2; but, since there are only three terms, it is clear that in whatever order they are placed there cannot be more than two variations of sign. The equation cannot therefore have more than two positive roots. *This is an analytical proof that there cannot be more than two real apsidal distances.*

423. If n is a fraction, say $n=p/q$ in its lowest terms, we write $u=w^q$; the indices of w are then integers and w and therefore u can have only two positive values. It is assumed that if q is an even integer the sign of F is given by some other considerations, for otherwise F would not be a one-valued function of u.

424. The propositions proved in Arts. 420 and 422 are not altogether the same. The complete curve found by integrating (A) may have several branches separated from each other so that the particle cannot pass from one to the other. In 420 it is proved that *the actual branch described* cannot have more than two unequal apsidal distances. In 422 it is proved that when $F=\mu u^n$ *all the branches together* cannot have more than two unequal apsidal distances.

If the force be some other one-valued function of the distance the complete curve may have more than two unequal apsidal distances.

425. *Ex.* 1. If $\left(\frac{du}{d\theta}\right)^2=A(u-a)(u-b)(u-c)$ be the differential equation of an orbit, prove that the central force is a one-valued function of the distance. Prove also that the curve has two branches and three unequal apsidal distances, and that either branch may be described if the initial conditions are suitable. See Arts. 309, 441.

Ex. 2. If the central force is $F=\mu u^n$, where $n>3$ and the velocity is greater than that from infinity, prove that the apsidal distances lie between p and q, where $2\mu=h^2(n-1)p^{n-3}$ and $h^2=Cq^2$. [This follows from a theorem in the theory of equations applied to equation (A) of Art. 422.]

426. The apsidal angle. *To find the apsidal angle when $F=\mu u^n$, where $n<3$, and the orbit is nearly circular.*

The equation of the path with these conditions has been found by continued approximation in Arts. 367 to 370.

Taking the first approximation, we see by referring to the equation (6) of those articles that $du/d\theta$ is zero only when $p\theta+\alpha=i\pi$, where i is any integer. These values of θ therefore determine the apses and the reciprocals of the two corresponding apsidal distances are $c(1\pm M)$. The apsidal angle described between two consecutive apses is therefore π/p, where $p^2=3-n$.

Taking the higher approximations, we use the equations (12) and (13) in the same way. The apsidal angle is therefore π/p, where

$$p=\sqrt{(3-n)}\,\{1-\tfrac{1}{24}(n-2)(n+1)M^2\}.$$

The reciprocals of the apsidal distances are very nearly $c(1\pm M)$.

427. There is another method of finding the apsidal angle which is founded on a direct integration of the equations of motion*. Beginning with

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu u^{n-2}}{h^2},$$

we have, as in Art. 422,

$$h^2\left(\frac{du}{d\theta}\right)^2=\frac{2\mu}{n-1}u^{n-1}-h^2u^2+C;$$

let $u=a$, $u=b$ be the reciprocals of the inner and outer apsidal distances. Since the right-hand side of the equation must vanish for each of these values of u, we have

$$\frac{2\mu}{n-1}a^{n-1}-h^2a^2+C=0,\qquad \frac{2\mu}{n-1}b^{n-1}-h^2b^2+C=0.$$

Eliminating h^2 and C we find

$$\left(\frac{d\theta}{du}\right)^2=\frac{a^{n-1}-b^{n-1}}{\Delta},\qquad \Delta=\begin{vmatrix}u^{n-1}, & u^2, & 1\\ a^{n-1}, & a^2, & 1\\ b^{n-1}, & b^2, & 1\end{vmatrix}$$

To find the apsidal angle we have to integrate the value of $d\theta$ from $u=b$ to a.

To simplify the limits we put $a=c(1+M)$, $b=c(1-M)$ and $u=c(1+Mx)$; the limits of integration are then $x=-1$ to $+1$. Also since the orbit is nearly circular, we suppose M to be a small quantity.

It now becomes necessary to expand Δ in powers of M. This may be effected by using some simple properties of determinants. If we subtract the upper row from each of the other two, the determinant is practically reduced to a determinant of two rows. Noticing that

$$\begin{aligned}(1\pm M)^{n-1}-(1+Mx)^{n-1}=-(n-1)M(x\mp1)\,\{1+CM(x\pm1)\\ +DM^2(x^2\pm x+1)+EM^3(x^3\pm x^2+x\pm1)+\&c.\},\end{aligned}$$

where $C=\frac{1}{2}(n-2)$, $D=\frac{1}{6}(n-2)(n-3)$, $E=\frac{1}{24}(n-2)(n-3)(n-4)$, we see that the new determinant is

$$\Delta=c^{n+1}M^2(n-1)(x^2-1)\begin{vmatrix}1+CM(x+1)+\&c., & 2+M(x+1)\\ 1+CM(x-1)+\&c., & 2+M(x-1)\end{vmatrix}.$$

Subtracting one row from the other and performing some evident simplifications, we find

$$\Delta=E^2(x^2-1)\{1+\tfrac{1}{3}(n-2)Mx+\tfrac{1}{12}(n-2)M^2((n-4)x^2+n-6)\},$$

where $E^2=2c^{n+1}M^3(n-1)(n-3)$. We thence deduce

$$\frac{E}{\sqrt{\Delta}}=\frac{1}{\sqrt{(x^2-1)}}\{1-\tfrac{1}{6}(n-2)Mx+\tfrac{1}{24}(n-2)M^2(2x^2-n+6)\}.$$

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* The method of finding the apsidal angle by a direct integration of the apsidal equation was first used by Bertrand, *Comptes Rendus*, vol. 77, 1873. An improved version was afterwards given by Darboux in his notes to the *Cours de Mécanique* by Despeyrous, 1886.

In the same way we find after some reductions

$$(a^{n-1}-b^{n-1})^{\frac{1}{2}}=\{2c^{n-1}M(n-1)\}^{\frac{1}{2}}\{1+\tfrac{1}{12}(n-2)(n-3)M^2\}.$$

Remembering that $du=cMdx$, these give

$$\frac{d\theta}{dx}=\frac{1}{\sqrt{(3-n)}}\frac{1}{\sqrt{(1-x^2)}}\left\{1-\frac{n-2}{6}Mx+\frac{n-2}{24}M^2(2x^2+n)\right\}.$$

The integrations can be effected at sight by putting $x=\sin\phi$. Taking the limits to be $\phi=\pm\frac{1}{2}\pi$ to make the apses adjacent, we find that the apsidal angle is

$$\frac{\pi}{\sqrt{(3-n)}}\left\{1+\frac{(n-2)(n+1)}{24}M^2\right\}.$$

428. **Closed orbits.** *An orbit is described about a centre of force whose attraction is a one-valued function of the distance. Prove that if the orbit is closed, for all initial conditions within certain defined limits, the law of force must be the inverse square or the direct distance.* [Bertrand, *Comptes Rendus*, vol. 77, 1873.]

If the path is closed and re-entering it must admit of both a maximum and a minimum radius vector. The orbit therefore has two apsidal distances and must lie between the two circles which have these for radii and their centres at the centre of force. By varying the initial conditions we may widen or diminish the space between the circles, yet by the question the orbit is always to be closed so long as the radii of the circles remain finite.

Representing the first approximation to the reciprocals of the radii by $c(1\pm M)$ the apsidal angle will be π/p, where p can be expressed in some series of ascending powers of M. The orbit cannot be closed unless the apsidal angle is such that, after some multiple of it has been described, the particle is again at the same point of space and moving in the same way. Hence *p must be a rational fraction for all values of M whether rational or not.* The coefficients of all the powers of M must therefore be zero, while the term independent of M must be a rational fraction.

When $F=\mu u^n$ the series for p is (Art. 426)

$$p=\sqrt{(3-n)}\{1-\tfrac{1}{24}(n-2)(n+1)M^2+\&c.\}.$$

Since the coefficient of M^2 must be zero we see that $n=2$ or -1, i.e. the law of force must be the inverse square or the direct distance. In either case the condition that $\sqrt{(3-n)}$ should be a rational fraction is satisfied.

If we take the most general form for the force, we have $F=u^2f(u)$. We know by Art. 368 that the first term of the

series for p is, in general, a function of c, i.e. of the reciprocal of the mean radius. Since this can be varied arbitrarily the apsidal angle cannot be commensurable with π unless this first term, viz. $cf'(c)/f(c)$, is independent of c. Putting this equal to a constant m we find by an easy integration that $f(c)=\mu c^m$. Hence $F=\mu u^{m+2}$. The general case is therefore reduced to the special case already considered.

429. Classification of orbits. *The force being $F=\mu u^n$ it is required to classify the various forms of the orbit according to the number of the apsidal distances*. We suppose μ to be positive and h not to be zero.*

Arranging the apsidal equation (A) (Art. 422) in descending powers of u, it takes one or other of the three following forms

$$\left(\frac{dr}{dt}\right)^2=h^2\left(\frac{du}{d\theta}\right)^2=\frac{2\mu}{n-1}u^{n-1}-h^2u^2+C \text{..........................(A)},$$

$$=-h^2u^2+\frac{2\mu}{n-1}u^{n-1}+C,$$

$$=\left\{-h^2u^{3-n}+Cu^{1-n}-\frac{2\mu}{1-n}\right\}u^{n-1},$$

according as $n>3$, n lies between 3 and 1, and $n<1$.

The two constants $\frac{1}{2}C$ and h determine the *energy* and *angular momentum* of the particle, Art. 313. When these are given, we arrive, by integrating (A), at an equation of the form $\theta+\alpha=f(u)$. By varying the constant α we turn the curve round the origin without altering its form. It follows that *when C and h are known, the orbit is determined in form but not in position.* The curve thus found may have several branches which are not connected with each other. *One point on the orbit must therefore also be given to determine the value of α and to distinguish the branch actually described by the particle.*

Any point on the curve being taken as the point of projection, we may regard v as the initial velocity. We thus have $C=v^2-V_1^2$ or $C=v^2+V_0^2$, where V_1 is the velocity from infinity, and V_0 the velocity to the origin. The first equation is to be used when V_1 is finite, i.e. when $n>1$; the second when V_0 is finite, i.e. when $n<1$. See Art. 313.

430. Case I. *Let the curve have but one apsidal distance.* The right-hand side of the apsidal equation (A) must change sign once as u varies from zero to infinity. Hence, when $n>3$, C is negative or zero, i.e. the velocity v is less than or equal to that from infinity; when n lies between 3 and 1, C must be positive or zero, i.e. the velocity v is greater than or equal to that from infinity. Lastly we see from the third form of the equation (A) that when $n<1$ the curve cannot have only one apsidal distance.

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* Korteweg, *Sur les trajectoires décrites sous l'influence d'une force centrale, Archives Néerlandaises*, vol. XIX. 1884, discusses the forms of the orbits, the conditions of stability and the asymptotic circles. Greenhill, *On the stability of orbits, Proc. Lond. Math. Soc.* vol. XXII. 1888, treats of the asymptotic circles which can be described when $F=\mu u^n$ for various values of n.

These conditions being satisfied, let $u=a$ be the reciprocal of the apsidal distance, found by solving the equation (A). We then have

$$\left(\frac{dr}{dt}\right)^2=h^2\left(\frac{du}{d\theta}\right)^2=(u-a)\,\phi(u),$$

where $\phi(u)$ cannot change sign as u varies from 0 to ∞. Since $\phi(u)$ must have the same sign as the highest power of u, its sign is positive or negative according as $n>$ or <3.

We notice that if n is a fraction, say $n=p/q$, we replace the factor $u-a$ by $w-b$ where $u=w^q$, $a=b^q$; Art. 423. As in most cases the force F varies as some integral power of the distance, it will be more convenient to retain the form given above.

Since the left-hand side of (2) is necessarily positive, the whole of the curve must lie inside the circle $u=a$ if $n>3$, and must lie outside that circle if $n<3$. Suppose the particle, as it moves round the centre of force, to have arrived at the apse. It will then begin to recede from the circle and must always continue to recede because $du/d\theta$ is not again zero. The orbit has therefore two branches extending from the apse to the centre of force or to infinity according as $n>$ or <3. The apse is an apocentre in the first case and a pericentre (as in a hyperbola described about the inner focus) in the second case.

The motion in the neighbourhood of the apse may be found by writing $u=a+x$ and retaining only the lowest powers of x. We then have

$$(dx/d\theta)^2=4Ax; \quad \therefore\ u-a=A\theta^2,$$

where $4Ah^2=\phi(a)$. The path is therefore such that the particle describes a *finite angle* θ while it moves from $u=u$ to $u=a$. Since $d\theta/dt=hu^2$ is finite, the time of describing this finite angle is also finite.

431. Cases II. and III. *To find the conditions that there may be either two apsidal distances or none.* The apsidal equation must have two positive roots or none. The condition for this is that the right-hand side of (A) must have the same sign when $u=0$ and $u=\infty$.

First. Let $n>3$, this condition requires that C should be positive and not zero. *The velocity at every point must therefore be greater than that from infinity.*

To distinguish the cases we find the max-min value M of the right-hand side by equating to zero its differential coefficient. We thus find

$$M=-\frac{\mu}{\kappa}\left(\frac{h^2}{\mu}\right)^\kappa+C, \qquad \kappa=\frac{n-1}{n-3}.$$

Taking the second differential coefficient we find that M is a minimum when $n>3$ and a maximum when $n<3$.

We notice that when $n>3$, the two terms of M have opposite signs and that we can make either predominate by giving h or C small values. Thus M may have any sign if the initial conditions are suitably chosen. *The path may therefore have either two apsidal distances or none; there will be two if M is negative and none if M is positive. If $M=0$ the apsidal distances are equal.*

Secondly, let $3>n>1$. The right-hand side of (A) cannot have the same sign when $u=0$ and $u=\infty$ unless C is negative. *The velocity at every point must therefore be less than that from infinity.*

Writing as before

$$M = \frac{\mu}{\kappa'}\left(\frac{\mu}{h^2}\right)^{\kappa'} + C, \qquad \kappa' = \frac{n-1}{3-n},$$

we shall prove that M is necessarily positive and has zero for its least value. Then since the right-hand side of (A) is negative when $u=0$ and $u=\infty$ and is equal to the positive quantity M for some intermediate value, *there must be two apsidal distances* which can be equal only when $M=0$.

To prove that M is positive, we notice that M is least when h is greatest. Since $h = vr\sin\beta$ (Art. 313) this occurs when $h=vr$, i.e. when the particle is projected perpendicularly to the radius vector. Substituting this value of h and remembering that $C = v^2 - V_1^2$, we can see by a simple differentiation that M is again least when $v^2 = \mu/r^{n-1}$, that is, when the velocity is equal to that in a circle. This value of v is less than the velocity from infinity (n being <3), and is therefore admissible here. Substituting this value of v we find that the minimum value of M is zero. The value of M is therefore positive and is zero only when the path is a circle.

We may also prove that the orbit has two apsidal distances by observing that since the velocity is insufficient to carry the particle to infinity, the orbit must have either an apocentre or must approach an asymptotic circle. In either case the apsidal equation has one positive root and therefore has another.

Thirdly, let $1>n$. Since $C = v^2 + V_0^2$ we notice that C must be positive. We now have

$$M = -\frac{\mu}{\kappa}\left(\frac{h^2}{\mu}\right)^{\kappa} + v^2 + V_0^2, \qquad \kappa = \frac{1-n}{3-n};$$

we may prove in the same way as before that M is least when $h=vr$ and $v^2 = \mu/r^{n-1}$ and that then $M = -\frac{2\mu}{1-n}r^{1-n} + V_0^2 = 0$ by Art. 312. Thus M is always positive and *the curve has two apsidal distances* which can be equal only in a circle.

We verify this result by noticing that since an infinite velocity is required to carry the particle to infinity (n being <1, Art. 312), the orbit must have an apocentre or approach an asymptotic circle. The apsidal equation must therefore have two positive roots.

432. It follows from what precedes that the curve defined by the apsidal equation (A) can be without an apse only when $n>3$. In that case the orbit extends from the centre of force to infinity.

We arrive at the same result by noticing that if there is no apse, *the velocity must be sufficient to carry the particle to infinity.* If $1>n$ this condition cannot be satisfied (Art. 312). If $n>1$ this condition requires C to be positive and it is evident that the second form of the apsidal equation has then a positive root.

It also follows that there can be an asymptotic circle only when $n>3$. For if the orbit be ultimately circular the constant M must be zero, and this cannot happen when $n<3$ unless the orbit is circular throughout. See also Art. 447.

433. *To find the motion when the orbit has two apsidal distances.* If a, b be the reciprocals of these distances, the apsidal equation (A) takes the form

$$h^2\left(\frac{du}{d\theta}\right)^2 = (u-a)(u-b)\,\phi(u),$$

where $\phi(u)$ is positive or negative according as $n >$ or < 3. Since the left-hand side is necessarily positive we see that u cannot lie between the limits a and b if $\phi(u)$ is positive but must lie between them if $\phi(u)$ is negative. The whole curve must therefore lie outside the annulus defined by the circles $u=a$, $u=b$ if $n>3$, and must lie within that annulus if $n<3$.

It appears that when $n>3$ the full curve defined by the differential equation (A) contains two distinct branches, either of which can be described by the particle with the given energy $\frac{1}{2}C$ and the given angular momentum h. These, being separated by the empty annulus, do not intersect, so that when the point of projection is given the particular branch described by the particle is determined. We notice also that this branch has only one apsidal distance though the complete curve has two.

When $n<3$ the path of the particle undulates between the two circles $u=a$, $u=b$, touching each alternately and being always concave to the centre of force.

434. Case IV. *To find the motion when the apsidal distances are equal.* The apsidal equation now takes the form

$$h^2 (du/d\theta)^2 = (u-a)^2 \phi(u).$$

The motion as the particle approaches the circle $u=a$ may be found by putting $u=a+x$ and retaining only the lowest powers of x. We then have

$$h^2 (dx/d\theta)^2 = \phi(a)\, x^2, \quad \therefore\ u-a=Ae^{-m\theta},$$

where $m^2=\phi(a)/h^2$. The particle therefore approaches the limiting circle in an asymptotic path and arrives at the circle only when $\theta=\infty$. Since $d\theta/dt$ (being ultimately equal to ha^2) is finite, the time of describing an infinite number of revolutions round the centre of force is infinite.

The conditions that the right-hand side of the apsidal equation (A) may have a square factor and be positive are (1) the coefficients of the highest and lowest powers must be positive, and (2) we must have $M=0$, Art. 431. If $n>3$, C must be positive, i.e. the *velocity at every point must be greater than that from infinity.* If $n<3$ the coefficient of the highest power of u is negative, and *there can be no asymptotic circle.* (See also Art. 432.)

435. When $n>3$ and it is known that the path has an apse, we may prove that *that apse is a pericentre or apocentre according as the velocity of projection is greater or less than the velocity in a circle at the same distance.* Let v be the velocity of the particle, V_2 the velocity in a circle at the same distance r, V_1 the velocity from infinity; then (Art. 313)

$$V_1^2 = \frac{2\mu}{n-1}\frac{1}{r^{n-1}}, \qquad V_2^2=\frac{\mu}{r^{n-1}}, \qquad v^2=V_1^2+C \quad\text{.................(1)},$$

$$\therefore\ v^2-V_2^2 = -\tfrac{1}{2}(n-3)\,V_1^2+C \quad\text{.............................(2)}.$$

If $r=r_1$ represent any apsidal distance, we have at that apse $v^2/\rho=F$, $V_2^2/r_1=F$. At a pericentre the orbit lies outside the circle of radius r_1, hence $\rho>r_1$ and $\therefore v^2>V_2^2$. At an apocentre the orbit lies inside the circle and $v^2<V_2^2$.

It follows by inspection of (2) that *at a pericentre* both sides of that equation are positive, and, since V_1 decreases when r increases, both sides must continue to be positive as the particle recedes from the origin. The particle also cannot arrive at a second apse, for this requires the left side to become negative. In the same way *at an apocentre* the two sides of (2) are negative and must continue to be negative as the particle approaches the origin. The conclusion is that *the velocity*

at any point is greater or less than that in a circle at the same distance according as the path has a pericentre or apocentre.

It follows also that *the path described cannot have both a pericentre and an apocentre.*

436. The following table sums up the possible orbits when $F=\mu u^n$.

$n>3$, $v \leqq V_1$	{one apsidal distance, path inside the circle.
$v>V_1$, M negative	{two apsidal distances, path inside or outside both circles according as v is $<$ or $>V_2$.
$v>V_1$, M positive	{no apsidal distance, the path extends from the centre of force to infinity.
$v>V_1$, $M=0$	{an asymptotic circle, approached from within or from without according as v is $<$ or $>V_2$.
$3>n>1$, $v \geqq V_1$	{one apsidal distance, path outside the circle.
$v<V_1$	{two apsidal distances, path between the circles.
$1>n$, $v<V_1$	{two apsidal distances, path between the circles.

Here V_2 is the velocity in a circle at the distance of the point of projection.

Ex. When the force $F=\mu u^n$ is repulsive show that the path, if not rectilinear, has a pericentre with branches stretching to infinity.

437. *The motion in the neighbourhood of the origin* is found by retaining the highest powers only of u. We thus have by (A), Art. 429,

$$\left(\frac{dr}{dt}\right)^2=h^2\left(\frac{du}{d\theta}\right)^2=B^2u^{n-1} \text{ or } -h^2u^2,$$

according as $n>3$ or <3, where $(n-1)B^2=2\mu$. The first alternative gives after integration, supposing the particle to be approaching the origin,

$$r^p-r_0{}^p=-\frac{Bp}{h}\theta, \qquad r^q-r_0{}^q=-Bqt,$$

where $p=\frac{1}{2}(n-3)$, $q=\frac{1}{2}(n+1)$; showing that the particle (except when $n=3$) describes a finite angle in a finite time when the radial distance decreases from $r=r_0$ to zero.

The negative sign in the second alternative shows that, when $n<3$, the particle cannot reach the origin unless $h=0$, i.e. unless the path is a radius vector.

438. *The motion at an infinite distance from the origin* is found by retaining the lowest powers only of u. We then have

$$\left(\frac{dr}{dt}\right)^2=h^2\left(\frac{du}{d\theta}\right)^2=C \text{ or } -\frac{2\mu}{1-n}u^{n-1},$$

according as $n>$ or <1. The negative sign in the second alternative shows that when $n<1$ the curve can have no branches which extend to infinity.

When C is positive, i.e. when the velocity v of projection is greater than that from infinity, the first alternative leads to

$$h(u-u_0)=-\theta\surd C, \qquad r-r_0=t\surd C,$$

showing that when the particle travels from $r=r_0$ to infinity it describes a finite angle θ round the origin, and that the time is infinite. The path therefore tends to a rectilinear asymptote whose distance from the origin is $-d\theta/du=h/\surd C$.

If however $C=0$, i.e. the velocity v of projection is equal to that from infinity, the lowest existing power of u in the apsidal equation (A) is u^2 or u^{n-1}. We

then have $$\left(\frac{dr}{dt}\right)^2 = h^2\left(\frac{du}{d\theta}\right)^2 = -h^2u^2, \text{ or } +\frac{2\mu}{n-1}u^{n-1},$$
according as $n>3$ or $n<3$ but >1. The first alternative shows that (except when $h=0$) there are no branches leading to infinity. The second alternative, i.e. $n<3$, gives, supposing the particle to recede from the origin,

$$r^p - r_0{}^p = \frac{Bp}{h}\theta, \qquad r^q - r_0{}^q = Bqt,$$

where $(n-1)B^2=2\mu$, $p=-\frac{1}{2}(3-n)$, $q=\frac{1}{2}(n+1)$. These equations show that as the particle proceeds from $r=r_0$ to infinity it describes a finite angle in an infinite time. The path tends to a rectilinear asymptote at an infinite distance from the origin.

439. Stability of the orbits. Referring to Art. 436 we see that when $n>3$ the orbit extends to the origin or to infinity except when the particle is approaching an asymptotic circle. The existence of such a circle depends on the equality of the factors of the right-hand side of the apsidal equation, and a slight change in the constants C, h may render the factors unequal or imaginary. In either case the new path will lead the particle either to the centre of force or to infinity. *Such orbits may be called unstable.*

When $n<3$ and the velocity of projection less than that from infinity, the path is restricted to lie between the two circles $u=a$, $u=b$, and the values of a and b depend on the constants C and h. Any slight disturbance will alter the values of these constants, but the orbit will still be restricted to lie between two circles though the radii will not be exactly the same as before. *Such orbits may be called stable.*

440. *Ex.* Prove that any small decrease of the angular momentum h or increase of the energy $\frac{1}{2}C$ will widen the annulus within which the particle moves; that is, will increase the oscillation of the particle on each side of the central line.

441. Apsidal boundaries when $F=f(u)$. When the law of force contains several terms the argument becomes more complicated. Let $F=\Sigma A_n u^n$, then

$$v^2 = h^2\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} = C + 2\Sigma\frac{A_n}{n-1}u^{n-1}.$$

Transposing the terms, the apsidal equation is

$$\left(\frac{dr}{dt}\right)^2 = h^2\left(\frac{du}{d\theta}\right)^2 = 2\Sigma\frac{A_n}{n-1}u^{n-1} - h^2u^2 + C \text{ .... (B)},$$
$$= (u-a_1)(u-a_2)\dots(u-a_\kappa)\,\phi(u),$$

where a_1, a_2, ... are positive quantities arranged in descending order, and $\phi(u)$ contains all the factors which do not vanish between $u=0$ and $u=\infty$. The factor $\phi(u)$ keeps one sign, viz. that of the highest power of u.

Let us divide the plane of motion into annular portions by circles whose common centre is at the centre of force and whose radii are the reciprocals of a_1, a_2, &c. Then since $(du/d\theta)^2$ changes sign when u passes any one of these boundaries, it is clear that the curve defined by the differential equation (B) can have branches only in the alternate annuli, the intervening ones being vacant. The space between $u=a_1$ and $u=\infty$ being occupied or vacant according as $\phi(u)$ is positive or negative.

If the initial position of the particle lie between any two contiguous circles, the subsequent path is restricted to lie between these circles and touches each alternately. If the initial position lie outside the greatest circle or inside the least, the subsequent path must also lie outside or inside these circles and must therefore extend to infinity or to the centre of force.

442. Next, let some of the factors of the apsidal equation be equal, say

$$\left(\frac{dr}{dt}\right)^2 = h^2\left(\frac{du}{d\theta}\right)^2 = (u-a)^m f(u)\,\phi(u),$$

where $f(u)$ has been written for the remaining factors. To determine the motion in the neighbourhood of the circle $u=a$, we write $u=a+x$ and retain only the lowest powers of x. We then have, supposing $m>2$,

$$\left(\frac{dx}{d\theta}\right)^2 = \frac{B^2}{h^2}x^m, \qquad \frac{1}{x^\kappa} - \frac{1}{x_0{}^\kappa} = \pm\frac{B\kappa}{h}\theta,$$

where $B^2=f(a)\,\phi(a)$, and $\kappa=\frac{1}{2}(m-2)$. The case in which $m=2$ is discussed in Art. 434. *We see that the circle $u=a$ is asymptotic.* The particle arrives at the circle after describing an infinite number of revolutions round the centre of force and at the end of an infinite time.

443. Let us trace the surface of revolution whose abscissa is r and ordinate $z=Fr^3$, and let the ordinate z be perpendicular to the plane of motion of the particle. We notice that *this surface is independent of the initial conditions and that its form depends solely on the law of force.*

It is easy to see that the ordinate z corresponding to any value of r represents the square of the angular momentum in a circular orbit described with radius r. It will therefore be useful also to trace the plane whose ordinate is $z=h^2$, where h is the angular momentum of the path described.

By describing circles whose radii are the abscissæ of the maximum and minimum ordinates of the surface, we may divide the plane of motion into annular portions in which the function $z=Fr^3$ is alternately increasing or decreasing *outwards* from the centre of force. These we may call the *ascending or descending portions of the surface.*

444. If r represent any apsidal distance, we have at the corresponding apse $v^2/\rho=F$ and $v=h/r$; hence $h^2=F\rho r^2$. At a pericentre the orbit lies outside the circle of radius r, hence $\rho>r$, and the angular momentum h of the path must be greater than that in a circle of radius r. In the same way, at an apocentre the orbit lies inside the circle, and the angular momentum h is less than that in a circle of radius r.

Referring to the surface $z=Fr^3$, we see that *a pericentral distance $r=OA$ must have an ordinate AA' less than that of the plane $z=h^2$*, and *an apocentral distance OB must have an ordinate BB' greater than that of the plane.* It immediately follows that if A, B are the pericentre and apocentre of *the same path, both the points A', B', cannot lie on the same descending portion of the surface.* This conclusion does not apply if A, B are the pericentre and apocentre of different branches of the complete curve; (Art. 441).

We infer from this result that an annular space on the plane of motion (Art. 443) in which Fr^3 decreases outwards has this element of instability, viz. that *a path having both a pericentre and an apocentre cannot be described within the space.* If the path have a pericentre the particle will leave the space on its outer margin;

if an apocentre it will move out of the space on its inner boundary. We see also that when the particle has left the annular space it must proceed to infinity or to the centre of force, unless it come into some other external annular space in which Fr^3 has increased sufficiently to exceed the h^2 of its own path or into some internal space in which Fr^3 has become less than h^2.

445. We may also deduce this result very simply from the radial resolution.

We have
$$\frac{d^2r}{dt^2}=r\left(\frac{d\theta}{dt}\right)^2-F=\frac{1}{r^3}(h^2-Fr^3).$$

As the particle approaches and passes an apocentre r increases to a maximum and decreases, hence dr/dt changes sign from positive to negative and d^2r/dt^2 is negative. In the same way, when the particle passes a pericentre, d^2r/dt^2 is positive. It immediately follows that at an apocentre $Fr^3>h^2$ and at a pericentre $Fr^3<h^2$.

446. If the orbit have an asymptotic circle $r=a$, the angular momentum h must be equal to that in a circle of that radius. Hence *the asymptotic circle must be the projection of some one of the intersections of the surface* $z=Fr^3$ *with the plane* $z=h^2$; (Art. 443).

As the asymptotic circle is itself an apocentre or pericentre, it follows, as in Art. 445, that when the particle is approaching the circle from within h^2-Fr^3 is negative and ultimately zero. Hence Fr^3 is decreasing *outwards*. When the particle is approaching the circle from without h^2-Fr^3 is positive and ultimately zero, hence Fr^3 is increasing *inwards*. In either case it follows that *only those intersections which lie on a descending portion of the surface* $z=Fr^3$ *can correspond to asymptotic circles*.

As each descending portion of the surface can have only one intersection with the plane $z=h^2$, there cannot be more asymptotic circles than descending branches.

There may be fewer asymptotic circles than descending branches because two conditions are necessary that an asymptotic circle of given radius $r=a$ should exist; (1) the angular momentum must be equal to that in the circle, and (2) the constant C must be such that the velocity at a distance $r=a$ is equal to that in the circle, i.e. $v^2/a=F$.

447. As an example, consider the force $F=\mu u^n$. If $n>3$, the surface $z=Fr^3$ has only a descending portion, there can therefore be one and only one asymptotic circle. Also the path described cannot have both an apocentre and a pericentre, though different branches of the same curve may have one an apocentre and another a pericentre. See Arts. 444, 446, 436. If $n<3$, the surface $z=Fr^3$ has only an ascending portion. Hence there cannot be an asymptotic circle, but the path can have both an apocentre and a pericentre.

448. *Ex.* Discuss the properties of the surface $Z=Fr-v^2$, where the velocity v is a known function of r given in Art. 441. Prove that (1) the abscissæ of its max-min ordinates are the same as those of the surface $z=Fr^3$, so that the ascending and descending portions of each correspond (Art. 443); (2) each asymptotic circle must be one of the intersections of the surface with the plane of motion; (3) conversely, if at any intersection we also have $z=h^2$, that intersection is an asymptotic circle.

The first result follows from $\dfrac{dZ}{dr}=\dfrac{1}{r^2}\dfrac{dz}{dr}$. To prove the second and third we

notice that when $Z=0$, the velocity is equal to that in a circle; and when $z=h^2$, the angular momentum is equal to that in a circle.

449. Examples. *Ex.* 1. Find the law of force with the lowest index of u such that an orbit can be described having two given asymptotic circles whose radii are the reciprocals of a and b, and find the path. Find also the conditions of projection that the path may be described.

Referring to Art. 441 we see that the right-hand side of the apsidal equation (B) must be $\mu(u-a)^2(u-b)^2$. We then find

$$F=\mu u^2(u-a)(u-b)(2u-a-b)+\mu' u^3,$$

and the angular momentum at projection must be $\sqrt{\mu'}$.

Ex. 2. Let $F=\mu u^2\{(u-a)(3u-a-b)+cu\}$, where F is the central force. If the conditions of projection are such that $h^2=\mu c$ and the velocity v when $u=a$ is $v^2=\mu ca^2$, show that the path is $\dfrac{a-b}{u-b}=(\tanh\theta)^2$, where $c\kappa^2=2(a-b)$. Show also that the curve has two infinite branches tending to the same asymptotic circle $u=a$, with an apse at a distance $1/b$.

Ex. 3. A particle arrives at an apse distant r from the centre of force with a velocity v equal to that in a circle at the distance r. If the velocity be reversed in direction, will the particle describe the same path in a reverse order or will it travel along the circle? See Art. 419.

At such an apse the radius of curvature ρ of the path must be equal to r. But since $\dfrac{1}{\rho}=u+\dfrac{d^2u}{d\theta^2}$ at any apse this requires that $d^2u/d\theta^2=0$. The apsidal equation (B) of Art. 441 must therefore have equal roots, and the apse is at the extremity of a path with an asymptotic circle. The particle therefore can never arrive at such an apse in any finite time (Art. 442).

If the particle be projected from a point on the asymptotic circle with the given values of v and h it may be said to describe either orbit, for the deviation of one from the other is indefinitely small at the end of any finite time.

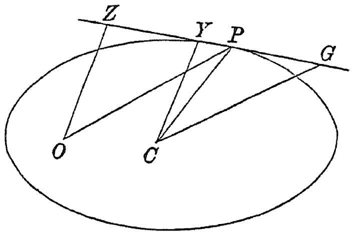
Boussinesq, *Comptes Rendus*, vol. 84, 1877, considers the circular motion to be a singular integral of the differential equation. Korteweg and Greenhill have also discussed this problem.

On the law of force by which a conic is described.

450. Newton's theorem*. *An orbit is described by a particle about a centre of force C whose law is known: it is required to find the law of force by which the same orbit can be described about another centre of force O.*

* Newton's theorem is given in Prop. VII. Cor. 3 of the second section of the first book of the *Principia*. The application to the motion of a particle in a circle acted on by a force parallel to a fixed direction follows in the next proposition. Sir W. R. Hamilton's paper, giving the law $F=\mu r/p^3$, is in the third volume of the *Proceedings of the Irish Academy*, 1846. Villarceau in the *Connaissance des Temps*

Let F, F' be the forces of attraction tending respectively to C and O. Let CY, OZ be the perpendiculars on the tangents at any point P; $CP=r$, $OP=r'$. Then since $\sin CPY = CY/r$, we have



$$\frac{v^2}{\rho} = F\frac{CY}{r}, \quad v = \frac{h}{CY},$$

$$\therefore\ F = \frac{h^2 r}{\rho \,.\, CY^3}.$$

Similarly $$F' = \frac{h'^2 r'}{\rho \,.\, OZ^3}, \qquad \therefore\ \frac{F'}{F} = \frac{h'^2}{h^2}\frac{r'}{r}\left(\frac{CY}{OZ}\right)^3.$$

If we draw CG parallel to OP, the triangles OPZ and CGY are similar, and

$$\frac{OZ}{OP} = \frac{CY}{CG}; \qquad \therefore\ \frac{F'}{F} = \frac{h'^2}{h^2}\frac{CG^3}{r'^2 r}.$$

If then F is given as a function of r, the law of force F' tending to any assumed point O is also known, when we have deduced CG as a function of r and r' from the geometrical properties of the curve.

Remembering that the area $A = \frac{1}{2}ht$, we see that the periodic times in which the whole curve is described about C and O respectively are inversely as the arbitrary constants h and h'. By choosing these properly we can make *the ratio of the periodic times have any ratio we please.*

We also notice that if the time of describing any arc PQ is known when the central force tends to C, the area PCQ is known. Now the area POQ differs from this by a *rectilinear*

for 1852, using Cartesian coordinates, arrived at two possible laws of force. Afterwards Darboux and Halphen investigated two laws equivalent to these, and proved that there is no other law in which the central force is a function only of the coordinates of its point of application. Their results may be found in vol. 84 of the *Comptes Rendus*, 1877. The investigations of Darboux were reproduced by him at somewhat greater length in his notes to the *Cours de Mécanique* by Despeyrous, 1884. There is a third paper by Glaisher in vol. 39 of the *Monthly Notices of the Astronomical Society*, 1878, who also gives the expression $(2\pi/\sqrt{\mu})\,\varpi^{\frac{3}{2}}$ for the periodic time. Darboux uses chiefly polar coordinates, while Halphen employs Cartesian, beginning with the general differential equation of all conics: Glaisher simplifies the arguments by frequently using geometrical methods. There is also a paper by S. Hirayama of Tokyo in *Gould's Astronomical Journal*, 1889.

figure whose area can therefore be found. Hence the area POQ and therefore the time of describing the same arc PQ when the central force tends to O can be found.

451. Suppose *the orbit is a conic,* then the force tending to the centre C is $F=\mu r$, and $h=\sqrt{\mu}\,.\,ab$. It immediately follows that *the force tending to any point O is* $F'=\dfrac{h'^2}{a^2b^2}\cdot\dfrac{CG^3}{r'^2}$. If, for example, O is a focus, it is a known geometrical property of a conic that G lies on the auxiliary circle and that therefore $CG=a$. We then have $F'=\mu'/r'^2$, where $h'^2=\mu'b^2/a$.

452. Parallel forces. *To find the force parallel to a given straight line by which a conic can be described.* See Art. 323.

Let the point O be at an infinite distance, then in Newton's formula PO and CG remain parallel to the given straight line throughout the motion. Also the length $r'=OP$ is constant. The required law of force is therefore $F'=\mu\,.\,CG^3$, where μ is some constant.

If the direction PO of the force at P cut the diameter conjugate to CG in N, we have $CG\,.\,PN=b'^2$, where b' is the semi-diameter parallel to CG. The law of force may therefore also be written $F'=A/PN^3$, where $A=\mu b'^6$.

To find the constant μ, we notice that in any central orbit, the velocity being $v=h/p$, the component of the velocity perpendicular to the radius vector r' is h/r'. In our case when the force acts parallel to a given straight line this component is constant. Representing this transverse velocity by V, the Newtonian formula of Art. 451 becomes $F'=\dfrac{V^2}{a^2b^2}CG^3$.

453. Hamilton's formula. *A particle describes a conic about a centre of force situated at any point O. It is required to find the law of force.* Taking the same notation as in Newton's theorem, we let F, F' be the forces tending respectively to the centre C and the point O. Then (Art. 450)

$$\frac{F'}{F}=\frac{h'^2}{h^2}\frac{OP}{CP}\left(\frac{CY}{OZ}\right)^3, \qquad F=\mu\,.\,CP, \qquad h=\sqrt{\mu}\,.\,ab.$$

It is a geometrical property of a conic that, if p and ϖ are the perpendiculars drawn from P and the centre C on the polar line

of O, $\dfrac{p}{\varpi} = \dfrac{OZ}{CY}$*. It follows that *the law of force tending to O is* $F' = \dfrac{h'^2}{a^2b^2}\left(\dfrac{\varpi}{p}\right)^3 r'$, where p and r' vary from point to point of the curve and h', a, b and ϖ are constant.

If we write the Hamiltonian expression for the force in the form $F' = \mu' r'/p^3$, we see that the angular momentum $h' = \sqrt{\mu'} . ab/\varpi^{\frac{3}{2}}$, where as before ϖ is the perpendicular from the centre on the polar line.

From this we easily deduce the periodic time in an elliptic orbit. Remembering that the whole area is πab, the formula $A = \frac{1}{2}h't$ gives as the time of describing a complete ellipse $T = \dfrac{2\pi}{\sqrt{\mu}} \varpi^{\frac{3}{2}}$.

454. *To find the time of describing any portion of the ellipse with Hamilton's law of force.* The coordinates of any point P referred to an origin at the centre of force O with axes parallel to the principal diameters are

$$x = a\cos\phi - f, \qquad y = b\sin\phi - g,$$

where ϕ is the eccentric angle of P and f, g the coordinates of the centre of force referred to the centre of the curve. Then, if h be the angular momentum,

$$hdt = xdy - ydx = (ab - fb\cos\phi - ga\sin\phi)\,d\phi,$$

$$\therefore\ ht = ab\phi - fb\sin\phi + ga\cos\phi - ga,$$

where the time is measured from the passage through the apse from which ϕ is measured. This, if required, can be expressed in terms of x and y,

$$ht = ab\phi - fy + gx - ga.$$

This result can be deduced at once from the formula $A = \frac{1}{2}ht$, by equating A to the excess of the area of the sector ACP (viz. $\frac{1}{2}ab\phi$) over the sum of the triangles ACO, OCP.

* The following is a short analytical proof: Let the conic be $Ax^2 + By^2 = 1$ and let f, g be the coordinates of O. The polar line of O and the tangent at P are respectively $\qquad Af\xi + Bg\eta = 1, \qquad Ax\xi + By\eta = 1.$

The perpendiculars from P and O, viz. p and OZ, are therefore

$$p = \frac{1 - Afx - Bgy}{\sqrt{(A^2f^2 + B^2g^2)}}, \qquad OZ = \frac{1 - Afx - Bgy}{\sqrt{(Ax^2 + By^2)}}.$$

The perpendiculars from the centre, viz. ϖ and CY, are found by replacing the numerators by unity. It follows that $p/OZ = \varpi/CY$.

The time of describing an arc of a hyperbola or parabola may be found by proceeding as in Arts. 348, 349.

455. Examples. *Ex.* 1. Deduce from Hamilton's expression (1) the central force to the focus of a conic, and (2) that to the centre. [In the latter case ϖ and p are both infinite but their ratio is unity.]

Ex. 2. A particle describes an ellipse whose centre is C under the action of a centre of force F situated at a point R in the major axis. If the tangent at P cut the major axis in T, prove that the force F varies as $RP \,.\, (CT/RT)^3$.

456. The Hamiltonian expression for the force may be put into two different forms.

First, we have the form $F=\mu r/p^3$ (Art. 453).

Secondly. Let OA, OB be two tangents drawn to the conic from the centre of force O, and let $PL=\alpha$, $PM=\beta$, $PN=\gamma$; these being the three perpendiculars drawn from any point P on the sides of the triangle OAB. By a property of conics we have $\alpha\beta=\kappa\gamma^2$, where κ is a constant for the same conic. The central force may therefore be expressed in either of the forms

$$F=\mu\frac{OP}{PN^3}=\nu\frac{OP}{(PL\,.\,PM)^{\frac{3}{2}}}. \qquad [\nu=\mu\kappa^{\frac{3}{2}}.]$$

Each of these expressions is a one-valued function of the position of P though their values are not necessarily equal except at points on the orbit.

We may suppose either of these laws to be extended to all points of the plane of motion and enquire what would be the path for any given conditions of projection. These problems will be considered in turn.

457. The conic being given in its general form referred to any rectangular axes, viz.,

$$Ax^2+2Cxy+By^2+2Dx+2Ey+G=0,$$

the two Hamiltonian expressions for the force to the origin may be put into the forms

$$F=\frac{h^2\Delta r}{(Dx+Ey+G)^3}, \qquad F=\frac{h^2\Delta r}{(\alpha x^2+2\gamma xy+\beta y^2)^{\frac{3}{2}}},$$

where $\alpha=D^2-AG$, $\gamma=DE-CG$, $\beta=E^2-BG$, and Δ is the discriminant.

To prove this we notice that the polar line of the origin is $Dx+Ey+G=0$, so that the ratio of the perpendiculars from the centre $\bar{x}$, $\bar{y}$ and from the point P is

$$\frac{\varpi}{p}=\frac{D\bar{x}+E\bar{y}+G}{Dx+Ey+G}.$$

If we refer the equation of the conic to the centre as origin, it becomes

$$Ax^2+2Cxy+By^2=-D\bar{x}-E\bar{y}-G=-\frac{\Delta}{AB-C^2}.$$

Turning the axes round the origin let this become

$$A'x^2+B'y^2=-D\bar{x}-E\bar{y}-G,$$

where by the theory of invariants $A'B'=AB-C^2$ and $A'+B'=A+B$. Since the conic is now referred to its principal diameters, we have $a^2b^2=\dfrac{(D\bar{x}+E\bar{y}+G)^2}{A'B'}$.

It immediately follows by substituting in Art. 453 that

$$F=\frac{h^2}{a^2b^2}\left(\frac{\varpi}{p}\right)^3 r=h^2\Delta\cdot\frac{r}{(Dx+Ey+G)^3}.$$

Since the equation of the conic may be written in the form

$$G\,(Ax^2+2Cxy+By^2)+(Dx+Ey+G)^2=(Dx+Ey)^2,$$

the expression just obtained for the force F may be put by a simple substitution into the second form.

The straight lines $\alpha x^2+2\gamma xy+\beta y^2=0$, when real, pass through the origin and make $Dx+Ey+G=0$. They therefore meet the curve at the points where the polar line of the origin cuts it, i.e. *these straight lines are the tangents drawn from the centre of force to the conic.*

458. In the same way we may express F as a function of the coordinates x, y in a variety of different forms each of which gives the same magnitude for the force when the particle lies on the given conic. When these expressions for the force are generalized and supposed to hold at all points of space, they are not always one-valued functions of the coordinates. A law which gives several different values for the force at the same point may be set aside as altogether improbable.

For example, we might deduce from Hamilton's law an expression for F in terms of r alone. To do this we find the distance p of any point P on the orbit from the polar line of the origin O in terms of the distance r of P from O. But there are four points on the conic at the same distance r from the origin and each of these is, in general, at a different distance from the polar line. The expression for the central force F as a function of r only will therefore have four values for each value of r.

459. The First law of force. Supposing the first form of the Hamiltonian law of force to be extended to all points of the plane, *we put* $F=\dfrac{\mu r}{p^3}$, *where r is the distance of any point P from a fixed centre of force O, and p is the perpendicular from P on an arbitrary straight line fixed in space.* It is supposed that p is positive when P and the origin are on the same side of the given straight line.

We shall now prove that, *if a particle be projected from any point P in any direction PT, with any velocity V, the path is a conic having O, and the given straight line, for pole and polar.*

This follows from the results of Art. 453. It is obvious that we can describe a conic to satisfy (1) the three conditions that it shall pass through P, touch PT and have such a radius of curvature that V^2/ρ is equal to the normal force at P, (2) the two conditions that the polar line of O shall be the given straight line. We may also prove that this conic is a *real conic.* This being so, the conic must be the path.

We may however obtain a proof independent of Art. 453 by integrating the equation of motion. Let the origin be at the centre of force, and the given straight line be parallel to the axis of x at a distance c, then $p=c-r\sin\theta$.

We have
$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}=\frac{\mu}{h^2}\frac{1}{(up)^3}=\frac{\mu}{h^2}\frac{1}{(cu-\sin\theta)^3}.$$

To integrate this, put $cu=\sin\theta+cu'$;
$$\therefore\ \frac{d^2u'}{d\theta^2}+u'=\frac{\mu}{h^2c^3u'^3}.$$

This is the differential equation of the path of a particle acted on by a central force $F=\mu r/c^3$. This path is known to be a conic having its centre at the origin, Art. 325;
$$\therefore\ c^2u'^2=A'\cos^2\theta+2C'\cos\theta\sin\theta+B'\sin^2\theta\ \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The polar equation of the required orbit is therefore
$$(cu-\sin\theta)^2=A'\cos^2\theta+2C'\cos\theta\sin\theta+B'\sin^2\theta,$$
which when written in Cartesian coordinates becomes
$$(c-y)^2=A'x^2+2C'xy+B'y^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Writing this equation in the form $\kappa\gamma^2=\alpha\beta$ where α, β are the factors of the right-hand side, it is obvious that the polar line of the origin is the given straight line $y=c$.

When the conic is given in the form (2), *the constant h is given by* $\frac{\mu c}{h^2}=A'B'-C'^2$.

To prove this we notice that h represents the angular momentum of both orbits. We have therefore by Art. 326 $h^2c^3/\mu=a'^2b'^2$, where a', b' are the semi-axes of the conic (1). We know by the theory of conics that $A'B'-C'^2=c^4/a'^2b'^2$, the result therefore follows at once.

When the conic is given in the general form of Art. 457, we find $\frac{\mu}{h^2}=\Delta\left(\frac{c}{G}\right)^3$.

Since the central force is not a function of r only, *it is not conservative* and the velocity cannot be found without a knowledge of the path. In such cases we use the formula $v=h/(OZ)$, see the figure of Art. 450.

460. *To classify the paths according to the sign of μ, the law of force being $F=\mu r/p^3$.*

Let μ be positive; the force is attractive and the orbit concave to O at all points on the side of the given straight line nearest to the centre of force and the contrary at all points on the far side. When a conic cuts the polar line of a point O, the part of the curve nearest to O is convex; hence *the orbit does not cut the polar line.* It also follows that the orbit may be an ellipse or hyperbola on the side near O, but must be a hyperbola on the far side.

Let μ be negative; the force is repulsive and the orbit convex to O on the near side of the polar line while the contrary holds on the far side. The conic may be an ellipse or a hyperbola. By drawing a figure we see that the polar line must cut the conic though, in the case of a hyperbola, the path may be the other branch.

461. Examples. *Ex.* 1. The conic $Ax^2+2Cxy+By^2+2cy-c^2=0$ is described by a particle under the action of a central force $F=\frac{\mu r}{p^3}$ tending to the origin, where $p=c-y$ is the distance of the particle from the given straight line

$y=c$. The conic must have the form given if the polar line of the origin is to be $y=c$. Prove that

(1) $\{A(B+1)-C^2\}h^2=\mu c$, (2) $\{AB-C^2\}\dfrac{h^2}{c^2}=\mu\dfrac{2p-c}{p^2}-\left(\dfrac{dy}{dt}\right)^2$,

(3) $Ah^2=\dfrac{\mu c}{p^2}y^2+\left(c\dfrac{dy}{dt}\right)^2$, (4) $(B+1)h^2=\dfrac{\mu c}{p^2}x^2+\left(c\dfrac{dx}{dt}+h\right)^2$,

(5) $-Ch^2=\dfrac{\mu c}{p^2}xy+\left(c\dfrac{dx}{dt}+h\right)c\dfrac{dy}{dt}$.

From these equations, when the path is known, we can find the angular momentum h and the two components of velocity; conversely we can deduce the path when the circumstances of projection are given.

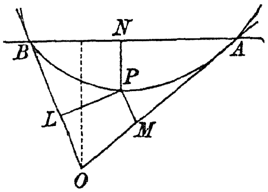
These equations follow from the preceding propositions. An independent proof may be obtained by differentiating the equation of the conic twice and writing for d^2x/dt^2, d^2y/dt^2 their values $-\mu x/p^3$, $-\mu y/p^3$. We thus obtain three equations which may be transformed into those given above by simple processes.

Ex. 2. Prove that the conic described is an ellipse, parabola or hyperbola according as $\mu(2p-c)/p^2-p'^2$ is positive, zero or negative, where p is the distance of the point of projection from the polar line and p' the *resolved* initial velocity.

Ex. 3. If $Ax^2+2Cxy+By^2+2Dx+2Ey+G=0$ is the conic described, show that the periodic time in an ellipse is $T=\dfrac{2\pi}{\sqrt{\mu}}\left\{\dfrac{\Delta c}{(AB-C^2)G}\right\}^{\frac{3}{2}}$.

Ex. 4. A particle is acted on by a central force $F=\mu\dfrac{r^{n-2}}{p^n}$ tending to the origin where r is the radius vector and p the distance from a fixed straight line. Prove that the equation of the path is $c/r=\sin\theta+f(\theta)$, where $c/r=f(\theta)$ is the polar equation of the path when the force tending to the origin is $F=\mu r^{n-2}/c^n$, both orbits being described with the same angular momentum h.

462. The second law of force. Supposing the second form of the Hamiltonian law of force to be extended to all points of the plane of motion, we put



$$F=\nu\frac{OP}{(PL\,.\,PM)^{\frac{3}{2}}},$$

where PL, PM are the perpendiculars from any point P on two fixed straight lines OA, OB, drawn through the centre of force O; Art. 456.

The form of the path may be obtained by following either of the methods described in Art. 459. The result is that the path is always a conic touching the given straight lines OA, OB.

If the force at any point P given by this formula is to be a function of the position of P only, it should be supposed to keep one sign throughout each of the triangular spaces formed by the given straight lines OA, OB (supposed to be real), though that sign may be different in different triangles. In any triangle in which the sign is negative only the convex portions of the conic can be described, while the concave portions are alone possible when the sign is positive. The force is infinite when the particle arrives at either of the straight lines OA, OB and the path becomes discontinuous.

If we suppose the magnitude alone of the force to be given by the formula, the

sign being taken at pleasure, arcs of both parts of each conic could be described by giving F the proper sign.

463. Examples. *Ex.* 1. If the trilinear equation of the conic is $\alpha\beta=\kappa\gamma^2$, prove that $h^2=-4\nu c\kappa^{\frac{1}{2}}\operatorname{cosec}^2\theta$ where $\nu=\mu\kappa^{\frac{3}{2}}$, θ is the angle at the corner of the triangle occupied by the central force, and c is the perpendicular from the centre of force O on the polar line AB. The negative sign shows (what is indeed obvious from the figure) that the force is repulsive on the side of the polar line nearest to the centre of force, i.e. μ is negative.

Ex. 2. A particle is projected from the point P with a velocity V and the tangent GPH intersects the given straight lines OA, OB in G and H. Prove that the areal equation of the path, referred to the triangle OGH, is

$$\sqrt{(lx)}+\sqrt{(my)}+\sqrt{\left(\frac{\rho\Delta}{em}z\right)}=0,$$

where $l=GP$, $m=HP$, Δ is the area of the triangle, and the radius of curvature ρ of the path at P is given by $V^2/\rho=F\sin GPO$. It follows that the conic is inscribed or escribed according as F is positive or negative, i.e. according as the force is attractive or repulsive.

464. There are no other laws of force besides

$$F=\mu\frac{OP}{PN^3}=\nu\frac{OP}{(PL\,.\,PM)^{\frac{3}{2}}},$$

which, being a one-valued function of the coordinates (except as regards sign), are such that a conic will be described with *any* initial conditions.

To prove this consider two conics intersecting in the four points A, B, C, D, which it is convenient to take as real. It follows from Hamilton's theorem that for points on any one conic the force to a given point O must be $F=\mu r/p^3$. Hence if the force is to be one-valued, i.e. the same at the same point of space for all paths through that point, we must have at each of the four points A, B, C, D, $p^3/\mu=\pm p'^3/\mu'$, where p, p' are the perpendiculars on the two polar lines of O.

We now require the following geometrical theorem*. If two conics intersect in four points A, B, C, D and the ratios of the perpendiculars from each of these points on the polar lines of a point O are equal, then either the polar lines are coincident or two common tangents (real or imaginary) can be drawn from O.

In the former case the common law of force for the two conics is given by the first form of F, in the latter case by the second form.

* Let the conics be, see Art. 457,

$$\alpha x^2+2\gamma xy+\beta y^2=(Dx+Ey+G)^2,$$
$$\alpha' x^2+2\gamma' xy+\beta' y^2=(D'x+E'y+G')^2.$$

Since $Dx+Ey+G=0$, $D'x+E'y+G'=0$ are the polar lines of the origin, we must have at the points of intersection

$$\alpha x^2+2\gamma xy+\beta y^2=m\,(\alpha' x^2+2\gamma' xy+\beta' y^2).$$

This quadratic equation gives only two values of y/x for the same value of m. The equation cannot therefore be satisfied at four points unless either α, β, γ are respectively proportional to α', β', γ', or the four points lie on two straight lines (say OAB, OCD) passing through O. In the former case the two conics have a pair of common tangents, in the latter the polar line of O is common to the two conics. This common polar line can be constructed by dividing OAB, OCD harmonically in E, F and then joining EF.

Singular Points in Central Orbits.

465. Singular Points. It has already been pointed out in Art. 100 that cases present themselves in our mathematical processes in which either the force, the velocity or both become infinite. Such infinite quantities do not occur in nature and if we limit ourselves to problems which have a direct application to natural phenomena these are only matters of curiosity. Nevertheless it is useful to consider them because they call our attention to peculiarities in the analysis which we might otherwise pass over. The utility of such a discussion is perhaps shown by *the differences of opinion* which exist regarding the subsequent path of a particle on arriving at a singular point*.

466. Points of infinite Force. Let us suppose that a particle P, describing an orbit about a centre of force O, arrives at a point B where the tangent passes through the centre of force and therefore coincides with the radius vector. At first sight we might suppose that the particle would move along the straight line BO and proceed in a direct line to the centre of force. But this is not necessarily the case.

Supposing B to be at a finite distance from O and the curvature to be finite, we see from the equations (Art. 306)

$$\frac{v^2}{\rho} = F\frac{p}{r}, \qquad v = \frac{h}{p}, \qquad r\frac{d\theta}{dt} = \frac{h}{r},$$

that both v and F are infinite at the point B. We shall also suppose that when the particle passes on *the force changes its direction* and reduces the velocity again to a finite quantity.

At the same time the component of the velocity perpendicular to the radius vector OP, viz. $r d\theta/dt$, remains finite however near the particle approaches B. Since there is no force to destroy this transverse velocity, the particle must cross the straight line OB and proceed to describe an arc on the opposite side.

* The singularity of the motion when the particle describes a circle about an external centre of force is discussed in Frost's *Newton*, 1854 and 1863. The same result is independently arrived at by Sylvester in the *Phil. Mag.* 1866. Other cases are considered by Asaph Hall in the *Messenger of Mathematics*, 1874. There are several papers also in the *Bulletin de la Société Mathématique de France*, such as Gascheau in vol. x. 1881, and Lecornu in vol. XXII.

467. To simplify the argument, let us suppose that the particle describes a circle about a centre of force O external to the circumference. By Art. 321, the circumstances of the motion are given by

$$F = \frac{\mu r}{(r^2 - b^2)^3}, \quad v^2 = \frac{\mu}{2}\frac{1}{(r^2 - b^2)^2}, \quad h^2 = \frac{\mu}{8a^2},$$

where b is the length of each of the tangents OB, OB' drawn from O to the circle.

Describe a second circle having a radius equal to that of the given circle and touching OB at B on the opposite side. If a second particle, properly projected along the second circle, arrive at B simultaneously with the given particle P, but moving in the opposite direction, both the velocity v and the transverse velocity h/r of the two particles will be equal and opposite each to each.

If the velocity of the second particle be reversed, Art. 419, it will retrace its former path in a reverse order and this must be also the subsequent path of the particle P.

The particle will therefore describe in succession a series of arcs of equal circles. The points of discontinuity at which the particle changes from one circle to the next lie on a circle whose centre is O and radius $OB = b$, and the successive arcs are alternately concave and convex to the centre of force. The particle will thus continually move round the centre of force *in the same direction* in an undulating orbit, but the curve will not be re-entering after one circuit unless the angle BOB' is a submultiple of four right angles.

The same arguments will apply to other orbits. When a conic is described about an external centre of force O as explained in Art. 462, the particle by a proper projection can be made to describe either of the arcs contained between the tangents drawn from O. On arriving at the point of contact B, it will cross the tangent and describe an arc of a conic equal to the undescribed arc of the original conic.

468. The particle arrives at the centre of force. When the particle P arrives at the centre of force in a finite time, the determination of the subsequent path presents some other peculiarities.

Taking first the Newtonian case in which the particle describes a circle about a centre of force O on its circumference, we notice that the transverse velocity h/r (as well as the velocity v) becomes infinite at O. To understand how the particle can

have an infinite velocity in a direction perpendicular to what is ultimately a tangent to the path, we observe that, since $2ap = r^2$, the transverse velocity h/r is infinitely less than the tangential velocity h/p.

When the particle has passed through the origin, the central force, changing its direction, reduces the velocity again to a finite quantity. Meantime the transverse velocity carries the particle across the tangent to the circle. By the same reasoning as before, the subsequent path is an equal circle which touches the original circle at the centre of force. On arriving a second time at the centre of force, the particle returns to the original circle, and so on continually.

469. One peculiarity of this case is that the radius vector of the particle while describing the second circle moves round the centre of force in the opposite direction to that in the first circle. Let P, P' be two positions of the particle, equidistant from the centre of force, just before and just after passing through that point. The transverse velocity being unaltered the moments of the velocity at P and P' taken in the same direction round O are equal and opposite. Since this moment is $r^2 d\theta/dt$, it follows that at the point of discontinuity h changes its sign.

470. When the particle moves in an equiangular spiral about a centre of force whose law is the inverse cube, it describes an infinite number of continually decreasing circuits and arrives at the centre of force at the end of a finite time, Art. 319. The subsequent path is another equiangular spiral, Art. 357, having the same angle. To determine its position we consider the conditions of motion at the point of junction.

Let us construct a second equiangular spiral obtained from the first by producing each radius vector PO backwards through the origin O to an equal distance OP'. If two particles P, P' describe these spirals so as to arrive simultaneously at the centre of force O, the particles are always in the same straight line with O, and at equal distances from it. Their radial and transverse velocities are also always equal and opposite each to each. If the velocity of P' be reversed, it will retrace its former path in a reverse order, and this must therefore be the subsequent path of P.

On passing the centre of force the particle will recede from the origin and describe the spiral above constructed. We notice also that the radius vector of the particle moves round the centre of force in the opposite direction to that in the first spiral.

471. Limiting Problems. We may sometimes simplify the discussion of some singularities by replacing the dynamical problem by another more general one of which the given problem is a limiting case. But the use of the method requires some discrimination. For example the motion of a particle attracted by a centre of force at a point O whose law of force is the inverse cube, may in some cases be regarded as a limit of the motion when the particle is constrained to move in a smooth fixed plane and is attracted by an equal centre of force situated at a point C outside the plane, where CO is perpendicular to the plane and is equal to some small quantity c. The method requires that the limiting motion should be the same whether we put the radius vector $r=0$ first and then $c=0$, or $c=0$ first and then $r=0$. We know by the principles of the differential calculus that the order in which the variables r and c assume their limiting values is not always a matter of indifference.

The component of force in the direction of the radius vector PO is $\mu r/(r^2+c^2)^2$ when the centre of force is at C, and is μ/r^3 when the centre is at O. As long as the particle is at a finite distance from the origin, these components are substantially the same, but when the particle is in the immediate neighbourhood of O, the former is $\mu r/c^4$ and therefore zero when the particle passes through O, while the latter is infinite.

In the former case, though the orbit at a distance from O is very nearly an equiangular spiral, it becomes elliptical in the neighbourhood of O. The force is not sufficient to draw the particle into the centre; the path has a pericentre and the particle retires again to an infinite distance. See also Art. 322.

472. Examples. *Ex.* 1. A particle describes one branch of the spiral $r\theta=a$ under the action of a centre of force in the origin (Art. 358). Show that after passing through the centre of force it will describe another spiral of the same kind, obtained from the first by producing each radius vector backwards through the origin to an equal distance.

Since the tangent to the curve is ultimately perpendicular to the radius vector, the two branches of the spiral may have a common tangent, and it might therefore be supposed that the particle would describe the second branch. But this argument requires that the particle should not pass through the origin, so that the radial velocity dr/dt (which is known to be constant) has its direction altered without any change in the direction of the force.

Ex. 2. A particle describes an epicycloid with the centre of force in the centre of the fixed circle (Art. 322). Supposing the force to become repulsive when the particle enters that circle, show that the path on passing the cusp is a hypocycloid.

Kepler's Problem.

473. *A particle describes an ellipse about a centre of force in one focus, it is required to express in series the two anomalies and the radius vector in terms of the time.*

If we require only the first few terms of the series it is convenient to start from the equations

$$r^2\frac{d\theta}{dt}=\sqrt{\{\mu a(1-e^2)\}},\quad \frac{a(1-e^2)}{r}=1+e\cos v\ \ldots\ldots(1),$$

where v is the true anomaly. Eliminating r, we have

$$\sqrt{\frac{\mu}{a^3}}\frac{dt}{d\theta}=(1-e^2)^{\frac{3}{2}}(1+e\cos v)^{-2}$$
$$=(1-\tfrac{3}{2}e^2+\&c.)(1-2e\cos v+3e^2\cos^2 v-\&c.)$$
$$=1-2e\cos v+\tfrac{3}{2}e^2\cos 2v+\&c.$$

Remembering that $v=\theta-\alpha$, where α is the longitude of the apse nearest to the centre of force, we have

$$nt+\epsilon=\theta-2e\sin(\theta-\alpha)+\tfrac{3}{4}e^2\sin 2(\theta-\alpha)+\&c.\ldots\ldots(2),$$

where $$n^2=\mu/a^3.$$

We notice that when the planet makes a complete revolution, θ increases by 2π and that the corresponding increment of t is $2\pi/n$. It follows immediately that n represents the mean angular velocity, the mean being taken with regard to the time; see Art. 341.

The equation (2) may be extended to higher powers of e, and therefore when e is small it may be used to determine the time of describing any angle θ.

474. To find θ in terms of t, we reverse the series. Writing it in the form

$$\theta = nt + \epsilon + 2e \sin(\theta - \alpha) - \tfrac{3}{4}e^2 \sin 2(\theta - \alpha),$$

we have as a first approximation

$$\theta = nt + \epsilon;$$

a second approximation gives

$$\theta = nt + \epsilon + 2e \sin(nt + \epsilon - \alpha).$$

Writing $v_0 = nt + \epsilon - \alpha$, a third approximation gives

$$\theta - \alpha = v_0 + 2e \sin(v_0 + 2e \sin v_0) - \tfrac{3}{4}e^2 \sin 2v_0;$$

$$\therefore\ \theta = nt + \epsilon + 2e \sin(nt + \epsilon - \alpha) + \tfrac{5}{4}e^2 \sin 2(nt + \epsilon - \alpha) \ldots (3),$$

and so on, the labour of effecting the successive approximations increasing at each step. As the eccentricity of the earth's orbit is about 1/60th it is obvious however that the terms become rapidly evanescent.

475. For the sake of clearness we recapitulate the meaning of the letters in the important equation we have just investigated; θ is the true longitude of the planet measured from any axis of x in the plane of the orbit; α is the longitude of the apse nearest the centre of force or origin; n is the mean angular velocity, the mean being taken with regard to time for one complete revolution; ϵ is a constant whose magnitude depends on the instant from which the time t is measured.

To define the epoch ϵ. Let a particle P_0 move round the centre of force in such a manner that its longitude is given by the equation $\theta_0 = nt + \epsilon$. It follows that this planet moves with a uniform angular velocity n and has therefore the same periodic time as the true planet P. When the radius vector of the particle P_0 passes through an apse $\theta_0 - \alpha$ and therefore $nt + \epsilon - \alpha$ is an

integral multiple of π. It immediately follows from (2) that $\theta = nt + \epsilon$. Hence the radii vectores of the two planets coincide when the true planet passes through either apse. The definition of P_0 may be shortly summed up thus.

Let an imaginary planet move round the centre of force with a uniform angular velocity in the same period as the true planet and let their radii vectores coincide at one apse and therefore at the other. This planet is called the Dynamical Mean Planet. Its longitude at the time $t = 0$ is the constant ϵ and is called the epoch.

476. *To express the mean anomaly and radius vector in terms of the time.*

Since both the mean and true planets cross the nearer apse at the time given by $nt + \epsilon = \alpha$, the mean anomaly may be represented by $m = nt + \epsilon$. If u be the eccentric anomaly we have by Art. 342,

$$u = m + e \sin u.$$

Proceeding as before we have for the three first approximations,

$$u = m, \qquad u = m + e \sin m,$$

$$\begin{aligned} u &= m + e \sin (m + e \sin m) \\ &= m + e \sin m + \tfrac{1}{2} e^2 \sin 2m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4). \end{aligned}$$

Again, as in Art. 343,

$$\begin{aligned} r &= a - ex = a - ae \cos u \\ &= a - ae \cos (m + e \sin m) \\ &= a \{1 - e \cos m + \tfrac{1}{2} e^2 (1 - \cos 2m)\} \quad \ldots\ldots\ldots (5). \end{aligned}$$

The series for the longitude and radius vector are given here only to the second power of the eccentricity. Laplace in the *Mécanique Céleste* (page 207) and Delaunay in his *Théorie de la Lune* (vol. I. pages 19 and 55) give the series up to the sixth power. Stone has continued the expansion up to the seventh power in the *Astronomical Notices*, 1896 (vol. LVI. page 110). Glaisher has given the expansion of the eccentric anomaly up to the eighth power in the *Astronomical Notices*, 1877 (vol. XXXVII. page 445).

477. *When the eccentricity e is very nearly equal to unity*, as in the case of some comets, the formulæ giving the relations between t and v must be modified. Starting as before (Art. 473) from the equations

$$r^2 \frac{d\theta}{dt} = \sqrt{\{\mu a (1 - e^2)\}}, \qquad \frac{a(1 - e^2)}{r} = 1 + e \cos v,$$

we put the perihelion distance $a(1 - e) = p$.

$$\therefore t \sqrt{\frac{\mu}{p^3}} = \int \frac{(1 + e)^{\frac{3}{2}} dv}{(1 + e \cos v)^2}.$$

Let $(1-e)/(1+e)=f$ and put $\tan\frac{1}{2}v=x$ for the sake of brevity;

$$\therefore \tfrac{1}{2}t\sqrt{\frac{\mu(1+e)}{p^3}}=\int\frac{(1+x^2)\,dx}{(1+fx^2)^2}$$

$$=x+\frac{x^3}{3}-2f\left(\frac{x^3}{3}+\frac{x^5}{5}\right)+3f^2\left(\frac{x^5}{5}+\frac{x^7}{7}\right)-\&c.$$

When v is given this formula determines the time t measured from perihelion. If f is small the term independent of f is the one requiring the most arithmetical calculation and this can be abbreviated by using the tables constructed for that purpose; see Art. 349. Conversely when t is given and v is required the same tables give a first approximate value of x. Representing this by $\tan\frac{1}{2}\omega$, it is usual to expand the correction $v-\omega$ in terms of ω in a series ascending in powers of f. For these formulæ we refer the reader to Watson's *Astronomy* and Gauss, *Theoria*, &c.

478. When the eccentric anomaly is given, the true and mean anomalies and the radius vector are expressed by the equations

$$m=u-e\sin u, \qquad \tan\frac{v}{2}=\sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2} \quad\ldots\ldots\ldots\ldots(1),$$

$$r=a-ex=a(1-e\cos u) \quad\ldots\ldots\ldots\ldots(2).$$

When any one of the other quantities is taken as the independent variable, the corresponding equations can be deduced from these in the form of series. *Two methods are used to find the general term of these series.* First we may have recourse to Lagrange's theorem, viz., when

$$y=z+x\phi(y), \qquad f(y)=f(z)+\Sigma\frac{x^i}{Li}\frac{d^{i-1}}{dz^{i-1}}\{(\phi(z))^i f'(z)\},$$

where $Li=1.2.3\ldots i$, and the Σ implies summation from $i=1$ to ∞. By the second method the general term is expressed by a definite integral which is usually a Bessel's function.

479. Lagrange's theorem. *To express the eccentric anomaly u and the radius vector r in terms of the time.*

Since $u=m+e\sin u$, we have by Lagrange's theorem

$$u=m+\Sigma\frac{e^i}{Li}\frac{d^{i-1}}{dm^{i-1}}(\sin m)^i.$$

The expansion of $(\sin m)^i$ in cosines of multiple angles when i is even and in sines when i is odd is given in books on trigonometry; (see Hobson's *Trigonometry*, Art. 52). The $(i-1)$th differential is always a series of sines and is easily seen to be

$$2^{i-1}\frac{d^{i-1}}{dm^{i-1}}(\sin m)^i=i^{i-1}\sin im-i(i-2)^{i-1}\sin(i-2)m+\frac{i(i-1)}{2}(i-4)^{i-1}\sin(i-4)m-\&c.$$

In the same way, expanding $\cos u$ by Lagrange's theorem, i.e. writing $f(y)=\cos y$, we find

$$\frac{r}{a}-1=-e\cos u=-e\cos m+\Sigma\frac{e^{i+1}}{Li}\frac{d^{i-1}}{dm^{i-1}}(\sin m)^{i+1},$$

where as before Σ implies summation from $i=1$ to ∞.

480. Bessel's functions. We shall now briefly examine the second method by which we express the general term in a definite integral. We know by Fourier's theorem that we can expand any function $\phi(m)$ in a series of the form

$$\phi(m)=A_0+A_1\cos m+\ldots+A_i\cos im+\ldots$$
$$+B_1\sin m+\ldots+B_i\sin im+\ldots,$$

which holds for all values of m from $-\pi$ to $+\pi$. If also $\phi(m)$ is a periodic function having the period 2π, the expansion will hold for all values of m. If $\phi(m)$ does not change sign with m we may omit the second line of the expansion, while if it does change sign with m, we omit the first line.

To find A_i we use Fourier's rule; multiply both sides by $\cos im$ and integrate from $m=-\pi$ to $+\pi$. Remembering that

$$\int \cos im \cos i'm\, dm=0, \qquad \int \cos im \sin i'm\, dm=0,$$
$$\int \cos^2 im\, dm=\int \sin^2 im\, dm=\pi,$$

we find

$$\int \phi(m) \cos im\, dm=\pi A_i, \qquad \int \phi(m)\, dm=2\pi A_0.$$

Similarly multiplying by $\sin im$ and integrating between the same limits, we find

$$\int \phi(m) \sin im\, dm=\pi B_i.$$

481. *To expand $u-m=e \sin u$ in a series of sines of multiples of m.* We put

$$u-m=\Sigma B_i \sin im;$$
$$\therefore\ \pi B_i=\int (u-m) \sin im\, dm,$$

the limits being $m=-\pi$ to π. Integrating by parts,

$$\pi i B_i=-(u-m)\cos im+\int \cos im\,(du-dm).$$

The integrated part is zero, for u and m are equal when $u=\pm\pi$. We thus have

$$\pi i B_i=\int \cos im\, du-\int \cos im\, dm.$$

The second integral is zero; substituting for m its value in terms of u,

$$\pi i B_i=\int \cos i\,(u-e \sin u)\, du.$$

This definite integral when taken between the limits 0 and π is written $\pi J_i(ie)$. We have

$$u=m+\Sigma B_i \sin im, \qquad iB_i=2J_i(ie).$$

482. The series thus obtained is convergent, for

$$\pi i^2 B_i=\int \frac{du}{dm}\, d \sin im=\frac{du}{dm} \sin im-\int \frac{d^2u}{dm^2} \sin im\, dm.$$

The integrated part vanishes at both the limits $m=\pm\pi$. Also

$$u=m+e \sin u, \qquad \therefore\ \frac{d^2u}{dm^2}=\frac{-e \sin u}{(1-e \cos u)^3},$$

and since $e<1$, it is clear that d^2u/dm^2 has a numerical maximum value; let this be k. Since $\sin im<1$, it follows that $\pi i^2 B_i$ is numerically $<2k\pi$. The series is therefore at least as convergent as $\Sigma 1/i^2$.

483. *To compare the two expansions of $u-m$.* In the Lagrangian series the terms are collected according to the powers of e, the coefficient of e^i being a series of the sines of multiple angles. In the series with Bessel's functions the terms are arranged according to the multiple angles, the coefficient of $\sin im$ being a series of powers of e.

The series for $u-m$ is really a double series containing both trigonometrical terms of the form $\sin im$ and also powers of e. If the terms are collected and arranged according to the multiple angles, it follows from what precedes, that each coefficient B_i is a convergent series, and that the series of coefficients B_1, B_2, &c. also form a convergent series, provided the eccentricity e is less than unity.

But if the series is arranged according to the powers of e, the positive and negative terms are added together in a different way. It may then be that the series of coefficients of e, e^2, &c. are only made convergent by more limited values of e. The condition of convergency is given in Art. 488.

484. The expression for B_i may be written

$$\pi i B_i = \int \cos iu \,.\, \{1 - \tfrac{1}{2} i^2 e^2 \sin^2 u + \&c.\}\, du + \int \sin iu \,.\, \{ie \sin u - \&c.\}\, du.$$

If we expand $\sin^2 u$, $\sin^4 u$, &c. in cosines of multiple angles and remember that $\int \cos iu \cos i'u\, du = 0$, we see that every term in the first integral will be zero in which the power of e is less than i. A similar remark applies to the second integral. Hence *the lowest power of e which accompanies the term sin im is* e^i.

485. *To express $r/a = 1 - e \cos u$ in a series of cosines of multiples of m, we put*

$$-e \cos u = A_0 + \Sigma A_i \cos im;$$

$$\therefore\ \pi A_i = -e \int \cos u \cos im\, dm,$$

where the limits of integration are $m = -\pi$ to π. Integrating by parts to change dm into du, we have

$$\pi i A_i = -e \cos u \sin im - e \int \sin im \sin u\, du.$$

The integrated part vanishes between the limits. Writing $m = u - e \sin u$, the integral becomes

$$\begin{aligned}\pi i A_i &= -e \int \sin i\,(u - e \sin u) \sin u\, du \\ &= \tfrac{1}{2} e \int \cos \{(i+1)\, u - ie \sin u\}\, du - \tfrac{1}{2} e \int \cos \{(i-1)\, u - ie \sin u\}\, du;\end{aligned}$$

$$\therefore\ iA_i = e\, \{J_{i+1}(ie) - J_{i-1}(ie)\}.$$

Similarly $\qquad 2\pi A_0 = -e \int \cos u\, dm = -e \int \cos u \,.\, (1 - e \cos u)\, du.$

Integrating between limits $u = -\pi$ to π, we find $A_0 = \tfrac{1}{2} e^2$:

$$\therefore\ r/a = 1 + \tfrac{1}{2} e^2 + \Sigma A_i \cos im.$$

486. That this series is converg. ıt may be proved in the same way as before. We have

$$\pi i^2 A_i = -e \int \frac{d \cos u}{dm}\, d \cos im = e \int \frac{d^2 \cos u}{dm^2} \cos im\, dm,$$

by integrating by parts. Since $u = m + e \sin u$, we find by differentiation $\dfrac{d^2 \cos u}{dm^2} = \dfrac{e - \cos u}{(1 - e \cos u)^2}$. This has obviously a maximum value, say k. Then since $\cos im < 1$, $\pi i^2 A_i$ is numerically less than $2\pi k e$, and the series is at least as convergent as $\Sigma 1/i^2$.

487. Examples. *Ex.* 1. Prove $\cos \kappa u = \Sigma A_i \cos im$, $\sin \kappa u = \Sigma B_i \sin im$, where $iA_i = \kappa\, \{J_{i-\kappa}(ie) - J_{i+\kappa}(ie)\}$, $iB_i = \kappa\, \{J_{i-\kappa}(ie) + J_{i+\kappa}(ie)\}$ and κ is not equal to unity, and the summations extend from $i = 1$ to ∞. Also $J_{-n}(x) = (-1)^n J_n(x)$.

Since $J_{-n}(-x) = J_n(x)$, these series may be written

$$\frac{1}{\kappa} \cos \kappa u = \Sigma J_{i-\kappa}(ie) \frac{\cos im}{i}, \qquad \frac{1}{\kappa} \sin \kappa u = \Sigma J_{i-\kappa}(ie) \frac{\sin im}{i},$$

where Σ implies summation from $i = -\infty$ to $+\infty$, and the term $J_{i-\kappa}(ie)/i$, when $i = 0$, is $-\tfrac{1}{2} e$ or 0 according as κ is equal or unequal to unity (Art. 485).

Since the Cartesian coordinates, referred to the centre of the ellipse, are $x = a \cos u$, $y = b \sin u$, we deduce the expansions of these in terms of the mean anomaly by putting $\kappa = 1$.

Ex. 2. Prove that $a/r = 1 + 2\Sigma J_i(ie) \cos im$, where the summation extends from $i = 1$ to ∞.

This follows from $a/r = du/dm$; see Arts. 343, 481.

Ex. 3. Prove that $v = m + \Sigma C_i \sin im$, where

$$C_i = \frac{2\sqrt{(1-e^2)}}{\pi i}\int_0^\pi \frac{\cos i(u - e\sin u)}{1 - e\cos u}\,du.$$

Proceeding as before we find $\pi i C_i = \int_{-\pi}^{+\pi} \cos im\,(dv - dm)$; substituting for dv/du, the result follows. Also by integrating again by parts, we can prove that this series is at least as convergent as $\Sigma 1/i^2$. This integral is given by Poisson in the *Connaissance des Temps*, 1825, 1836. See also Laplace, vol. v. and Lefort, *Liouville's Journal*, 1846. See also Art. 343.

Ex. 4. Prove the expansions

$$\tfrac{1}{2}(v-u) = \lambda\sin u + \tfrac{1}{2}\lambda^2 \sin 2u + \tfrac{1}{3}\lambda^3 \sin 3u + \ldots$$
$$\tfrac{1}{2}(u-v) = -\lambda\sin v + \tfrac{1}{2}\lambda^2 \sin 2v - \tfrac{1}{3}\lambda^3 \sin 3v - \ldots$$

where $$\lambda = \frac{e}{1+\sqrt{(1-e^2)}}.$$ [Laplace.]

In $\tan\frac{1}{2}v = \mu\tan\frac{1}{2}u$, where $\mu^2 = (1+e)/(1-e)$, substitute the exponential values of the tangents, solve for $e^{(v-u)\sqrt{-1}}$ and take logarithms; the results follow easily.

Ex. 5. Show that $m = v + 2\Sigma\frac{(-\lambda)^i}{i}\{1 + i\sqrt{(1-e^2)}\}\sin iv$ where Σ implies summation from $i=1$ to ∞.

We have from the geometrical meaning of u, $r\sin v = b\sin u$ (Art. 342),

$$\therefore \sin u = \frac{\sqrt{(1-e^2)}\sin v}{1+e\cos v} = -\sqrt{(1-e^2)}\frac{d}{edv}\log(1+e\cos v)$$
$$= -\sqrt{(1-e^2)}\frac{d}{edv}\log\frac{(1+\lambda e^{v\sqrt{-1}})(1+\lambda e^{-v\sqrt{-1}})}{1+\lambda^2}.$$

Expand, substitute in $m = u - e\sin u$, remembering the theorem in Ex. 3, the result follows. This is Tisserand's proof of Laplace's theorem, *Méc. Céleste*, page 223.

488. Convergency of the series for r and θ. Laplace was the first to prove that the expansions of the radius vector and true anomaly in terms of the time and in powers of the eccentricity are not convergent for all values of the eccentricity less than unity (see Arts. 474, 476). He showed by a difficult and long process that the condition necessary for the convergence of both series is that the eccentricity should be less than ·66195. *Méc. Céleste*, Tome v. Supplément, p. 516.

This important result was afterwards confirmed by Cauchy, *Exercises d'Analyse*, &c. An account is also given by Moigno in his *Differential Calculus*. The whole argument was put on a better foundation by Rouché in a memoir on Lagrange's series in the *Journal Polytechnique*, Tome XXII. The process was afterwards further simplified by Hermite in his *Cours à la Faculté des Sciences*, Paris 1886. In these investigations the test of convergency requires the use of the complex variable. The latter part of the method of Rouché may be found in Tisserand, *Méc. Céleste*, Art. 100, and is also given here.

489. The theorem arrived at may be briefly stated. Having given the equation $z = m + x\phi(z)$ we have (1) to distinguish which root we expand in powers of x, (2) to determine the test of convergency. It is shown that if a contour exist enclosing the complex point $z = m$, such that at every point of the boundary the

modulus of $\frac{x\phi(z)}{z-m}$ is less than unity, the given equation has but one root within the area and the Lagrangian expansion for that root is convergent*.

To apply this theorem to Kepler's problem we put $\phi(z) = \sin z$ and let x represent the eccentricity of the ellipse, Art. 478.

We measure a real length $OA = m$ from an assumed origin O, and with A for centre describe a circle with an arbitrary radius r. Representing the complex line OP by z, the Lagrangian series will be convergent if r can be so chosen that the modulus of $\frac{x \sin z}{z-m}$ is less than unity for all positions of P on the circle. Since

$$(\text{mod})^2 \text{ of } \phi(\xi + \eta i) = \phi(\xi + \eta i) \cdot \phi(\xi - \eta i),$$

$$z = m + re^{\theta i},$$

where e is the base of Napier's logarithms, we have

$$(\text{mod})^2 \text{ of } \frac{x \sin z}{z-m} = \left(\frac{x}{r}\right)^2 \frac{\sin(m + re^{\theta i}) \sin(m + re^{-\theta i})}{e^{\theta i} e^{-\theta i}}$$

$$= \frac{1}{2}\left(\frac{x}{r}\right)^2 \{\cos(2ri \sin\theta) - \cos(2m + 2r\cos\theta)\}$$

$$= \left(\frac{x}{r}\right)^2 \{\tfrac{1}{4}(e^{r\sin\theta} + e^{-r\sin\theta})^2 - \cos^2(m + r\cos\theta)\}.$$

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* If $f(x)$ be a continuous one-valued function over the area of a circular contour whose centre is $x = a$, then Cauchy's theorem asserts that $f(x)$ can be expanded by Taylor's theorem in a convergent series of powers of $x - a$ for all points within the contour; (see Forsyth's *Theory of Functions*, Art. 26).

When $z = m + x\phi(z)$, the Lagrangian expansion of z, or $\psi(z)$, in powers of x is a transformation, term for term, of Taylor's, and we may use Cauchy's theorem, provided z, or $\psi(z)$, is one-valued.

If z have two values for the same value of x, the equation $F(z) = z - m - x\phi(z) = 0$ (regarded as an equation to find z when x is given) has two roots. To determine whether this is so, we use another theorem of Cauchy's (see Burnside and Panton, *Theory of Equations*).

We measure $OA = m$ from the assumed origin O and with A for centre describe a circle of radius r. Let a point P describe this circle once, then by Cauchy's theorem if $\log F(z)$ is increased by $2n\pi i$, the equation $F(z)$ has n roots within the contour. Hermite writes

$$\log F(z) = \log(z - m) + \log\left(1 - \frac{x\phi(z)}{z-m}\right).$$

(1) The equation $z - m = 0$ has but one root and that root lies within the contour, hence as P moves round, $\log(z - m)$ is increased by $2\pi i$.

(2) If the modulus of $u = \frac{x\phi(z)}{z-m}$ is less than unity at all points of the circle the value of $\log(1-u)$, (being the same on departing from and arriving again at any point of the contour) increases by zero when P moves round the contour.

It follows that $\log F(z)$ increases by $2\pi i$ when P makes one circuit, that is *the equation $z = m + x\phi(z)$ has but one root within the contour if the modulus of $\frac{x\phi(z)}{z-m}$ is less than unity at all points on the circumference.*

Now, putting $e^{r\sin\theta}+e^{-r\sin\theta}=v+\frac{1}{v}$, we see that the first term of this expression continually increases from $v=1$, or $\theta=0$ to $v=\infty$, and is therefore greatest when $\theta=\frac{1}{2}\pi$. The least value of the second term is zero. The modulus is therefore less than $\frac{1}{2}\frac{x}{r}(e^r+e^{-r})$. The Lagrangian series is therefore convergent for all values of the eccentricity x less than $2r/(e^r+e^{-r})$.

To find the maximum value of this function of r, we equate its differential coefficient to zero. This gives

$$V=e^r(r-1)-e^{-r}(r+1)=0.$$

Since dV/dr is positive for all values of r this equation has but one positive root, and this root lies between 1 and 2. Using the value of e^r given by the equation $V=0$, we find that the maximum value of the eccentricity is $\sqrt{(r^2-1)}$, which reduces to ·66.

CHAPTER VII.

MOTION IN THREE DIMENSIONS.

The four elementary resolutions and moving axes.

490. The Cartesian equations. The equations of motion of a particle in three dimensions may be written in a variety of forms all of which are much used.

The Cartesian forms of these equations are

$$\frac{d^2x}{dt^2}=X,\quad \frac{d^2y}{dt^2}=Y,\quad \frac{d^2z}{dt^2}=Z \text{ ....(A)},$$

where x, y, z are the coordinates of the particle and X, Y, Z the components of the accelerating forces on the particle. These equations are commonly used with rectangular axes, but it is obvious that they hold for oblique axes also, provided X, Y, Z are obtained by oblique resolution.

491. The Cylindrical equations. From these we may deduce *the cylindrical or semi-polar forms of the equations.* Let the coordinates of the particle P be ρ, ϕ, z, where ρ, ϕ are the polar coordinates in the plane of xy of the projection N of the particle P on that plane, and $z=PN$. By referring to Art. 35 we see that the first two of the equations (A) change by resolution into the first two of the following equations (B), while the third remains unaltered. We have

$$\frac{d^2\rho}{dt^2}-\rho\left(\frac{d\phi}{dt}\right)^2=P,\quad \frac{1}{\rho}\frac{d}{dt}\left(\rho^2\frac{d\phi}{dt}\right)=Q,\quad \frac{d^2z}{dt^2}=Z \text{ ....(B)},$$

where P, Q are the components of the accelerating forces respectively along and perpendicular to the radius vector ρ.

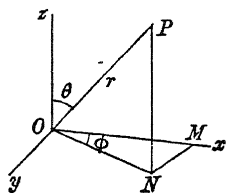
492. Principle of angular momentum. Since the moments of the components P and Z about the axis of z are zero, the moment of the whole acceleration about the axis of z is equal to $Q\rho$. In the same way the moment of the velocity about Oz is equal to the moment of its component perpendicular to the plane POz, and this is $\rho^2 d\phi/dt$. Introducing the mass m of the particle as a factor, the second of the equations (B) may be written in the form

$$\frac{d}{dt}\begin{pmatrix}\text{moment of}\\ \text{momentum}\end{pmatrix} = \begin{pmatrix}\text{moment of}\\ \text{forces}\end{pmatrix}.$$

The moments may be taken about any straight line which is fixed in space, such a line being here represented by the axis of z. The moment of the momentum is also called the angular momentum of the particle (Arts. 79, 260).

When the forces have no moment about a fixed straight line the angular momentum about that straight line is constant throughout the motion.

493. The polar equations. *We may immediately deduce from the semi-polar form* (B), *the polar equations* (C). Let r, θ, ϕ be the polar coordinates of P, where $r = OP$, θ is the angle OP makes with the axis Oz, and ϕ the angle the plane POz makes with the plane xOz.



Since $OP = r$ is the radius vector corresponding to the coordinates $ON = \rho$, $NP = z$, we see by Art. 35 that the accelerations

$$\frac{d^2\rho}{dt^2} \text{ and } \frac{d^2z}{dt^2} \text{ are equal to } \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \text{ and } \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right).$$

Hence the whole acceleration of P is the resultant of

(1) $\dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2$ along OP in the direction in which r is measured;

(2) $\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right)$ perpendicular to OP, in the plane zOP, taken positively in the direction in which θ is measured;

(3) $\rho\left(\frac{d\phi}{dt}\right)^2$ in the direction of the perpendicular drawn from P on Oz, i.e. parallel to NO;

(4) $\frac{1}{\rho}\frac{d}{dt}\left(\rho^2\frac{d\phi}{dt}\right)$ perpendicular to the plane zOP in the direction in which ϕ increases.

If R, S, T are the components of the acceleration of the particle respectively in the directions of (1) the radius vector OP, (2) the perpendicular to OP in the plane of zOP, and (3) the perpendicular to the plane zOP, taken positively when they act in the directions in which r, θ, ϕ are respectively increasing, we have

$$\left.\begin{aligned} \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - \rho\left(\frac{d\phi}{dt}\right)^2 \sin\theta &= R \\ \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) - \rho\left(\frac{d\phi}{dt}\right)^2 \cos\theta &= S \\ \frac{1}{\rho}\frac{d}{dt}\left(\rho^2\frac{d\phi}{dt}\right) \qquad\qquad &= T \end{aligned}\right\} \text{............(C).}$$

We notice that $\rho = r\sin\theta$.

494. *Ex.* If v be the velocity, show that the radial acceleration is

$$P = \frac{d^2r}{dt^2} + \frac{1}{r}\left\{\left(\frac{dr}{dt}\right)^2 - v^2\right\}.$$

495. Reducing a plane to rest. Referring to the semi-polar equations (B), we notice that if we transfer the term $\rho\,(d\phi/dt)^2$ to the right-hand side of the first equation and include it among the impressed accelerating forces, the first and third equations become the same as the Cartesian equations of motion of a particle moving in a fixed plane zOP (Art. 31), while the second equation determines the motion perpendicular to that plane. *We may therefore replace the first and third resolutions by any of the other forms which have been proved to be equivalent to them.* Art. 38.

For example, if we replace these two resolutions by their polar forms (Art. 35) we obtain at once the equations (C).

The process of regarding $\rho\,(d\phi/dt)^2$ as an impressed accelerating force acting at P and tending *from* the axis of z is sometimes called *reducing the plane zOP to rest.* See Arts. 197, 257.

496. The intrinsic equations. *To find the intrinsic equations of motion, due to the tangential and normal resolutions.*

Let P, P' be the positions of the particle at the times t, $t+dt$; v, $v+dv$ the velocities in those positions, $d\psi$ the angle between the tangents.

In the time dt, the component of velocity along the tangent at P has increased from v to $(v+dv)\cos d\psi$. Writing unity for $\cos d\psi$, the acceleration along the tangent, i.e. the rate of increase of the velocity, is dv/dt.

The component of velocity along the radius of curvature at P has increased from zero to $(v+dv)\sin d\psi$, which in the limit is $v\,d\psi$. The acceleration along the radius of curvature is therefore $v\,d\psi/dt$, or which is the same thing v^2/ρ.

The osculating plane by definition contains two consecutive tangents. The component of velocity perpendicular to that plane is zero and remains zero. The acceleration along the perpendicular to the osculating plane, i.e. the binormal, is therefore zero.

If F and G are the component accelerations measured positively in the directions of the arc s, the radius of curvature ρ and H the component perpendicular to the osculating plane, the equations of motion are

$$v\frac{dv}{ds}=F,\quad \frac{v^2}{\rho}=G,\quad 0=H\,\ldots\ldots\ldots\ldots\ldots(\text{D}).$$

497. *Show that the solution of the equations of motion of a particle in polar coordinates can be reduced to integrations when the work function has the form*

$$U=f_1(r)+\frac{f_2(\theta)}{r^2}+\frac{f_3(\phi)}{r^2\sin^2\theta},$$

where $f_1(r)$, $f_2(\theta)$ and $f_3(\phi)$ are arbitrary functions.

The third of the equations (C) gives, with this form of U, the mass being unity,

$$\frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2\sin^2\theta\frac{d\phi}{dt}\right)=\frac{df_3(\phi)}{r\sin\theta\,d\phi}\frac{1}{r^2\sin^2\theta};$$

$$\therefore\ \frac{1}{2}\left(r^2\sin^2\theta\frac{d\phi}{dt}\right)^2=f_3(\phi)+A\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The second of the equations (C) gives

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)-r\sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2=\frac{df_2(\theta)}{r\,d\theta}\frac{1}{r^2}-\frac{2f_3(\phi)\cos\theta}{r^3\sin^3\theta}.$$

Substituting for $d\phi/dt$, we obtain

$$\frac{1}{2}\left(r^2\frac{d\theta}{dt}\right)^2=-\frac{A}{\sin^2\theta}+f_2(\theta)+B\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The equation of vis viva is

$$\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2+r^2\sin^2\theta\left(\frac{d\phi}{dt}\right)^2=2U+2C \quad\ldots\ldots\ldots\ldots\ldots \quad (3).$$

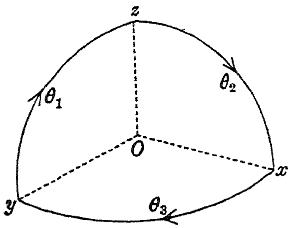
After substituting from (1) and (2) this becomes

$$\left(\frac{dr}{dt}\right)^2+\frac{2B}{r^2}=2f_1(r)+2C \quad\ldots\ldots\ldots\ldots\ldots \quad (4).$$

These are the first integrals of the equations of motion. Since the variables are separable in all the equations, they can be reduced to integrations. Substituting for dt from (4) in (2), that equation gives θ in terms of r. Substituting again in (1), we find ϕ in terms of r. Lastly (4) determines t in terms of r.

498. Moving axes. *To find the equations of motion of a particle referred to rectangular axes which move about the origin O in an arbitrary manner.*

Let us suppose that the moving axes Ox, Oy, Oz are turning round some instantaneous axis OI with an angular velocity which we may call θ. Let θ_1, θ_2, θ_3 be the components of θ about the instantaneous positions of Ox, Oy, Oz. Then in the figure θ_1 represents the rate at which any point in the circular arc yOz is moving along that arc, θ_2 is the rate at which any point of the circular arc zOx is moving along the arc, and so on.



Let us represent by the symbol V any directed quantity or vector such as a force, a velocity, or an acceleration. Let V_x, V_y, V_z be its components with regard to the moving axes.

Let $O\xi$, $O\eta$, $O\zeta$ be three rectangular axes fixed in space and let V_1, V_2, V_3 be the components of the same vector along these axes. Let α, β, γ be the angles the axis $O\zeta$ makes with Ox, Oy, Oz. Then

$$V_3=V_x\cos\alpha+V_y\cos\beta+V_z\cos\gamma,$$

$$\therefore\ \frac{dV_3}{dt}=\frac{dV_x}{dt}\cos\alpha+\frac{dV_y}{dt}\cos\beta+\frac{dV_z}{dt}\cos\gamma$$

$$-V_x\sin\alpha\frac{d\alpha}{dt}-V_y\sin\beta\frac{d\beta}{dt}-V_z\sin\gamma\frac{d\gamma}{dt}.$$

Let the arbitrary axis of ζ coincide with Oz at the time t, i.e. let the moving axis be passing through the fixed axis. Then $\alpha=\frac{1}{2}\pi$,

$\beta = \frac{1}{2}\pi$, $\gamma = 0$. Hence

$$\frac{dV_3}{dt} = \frac{dV_z}{dt} - V_x \frac{d\alpha}{dt} - V_y \frac{d\beta}{dt}.$$

Now $d\alpha/dt$ is the angular rate at which the axis Ox is separating from a fixed line $O\zeta$ momentarily coincident with Oz, hence $d\alpha/dt = \theta_2$. Similarly $d\beta/dt = -\theta_1$. Substituting

$$\frac{dV_3}{dt} = \frac{dV_z}{dt} - V_x\theta_2 + V_y\theta_1.$$

Similarly
$$\frac{dV_1}{dt} = \frac{dV_x}{dt} - V_y\theta_3 + V_z\theta_2,$$

$$\frac{dV_2}{dt} = \frac{dV_y}{dt} - V_z\theta_1 + V_x\theta_3.$$

When the moving axes momentarily coincide with the fixed axes, the components of the vector V are equal, each to each, i.e. $V_x = V_1$, $V_y = V_2$, $V_z = V_3$. As the moving axes pass on, this equality ceases to exist. The rates of increase of the components relatively to the moving axes are dV_x/dt, dV_y/dt, dV_z/dt; while the rates of increase relative to the fixed axes are dV_1/dt, dV_2/dt, dV_3/dt. The relations which exist between these rates of increase are given by the equations just investigated.

499. If the vector V is the radius vector of a moving point P, the components V_x, V_y, V_z are the Cartesian coordinates of P, and the rates of increase are the component velocities. If the vector V is the velocity of P, the rates of increase are the component accelerations.

Let then x, y, z be the coordinates of a point P; u, v, w the components of its velocity in space; X, Y, Z the components of its accelerations. Then

$$u = \frac{dx}{dt} - y\theta_3 + z\theta_2, \qquad X = \frac{du}{dt} - v\theta_3 + w\theta_2,$$

$$v = \frac{dy}{dt} - z\theta_1 + x\theta_3, \qquad Y = \frac{dv}{dt} - w\theta_1 + u\theta_3,$$

$$w = \frac{dz}{dt} - x\theta_2 + y\theta_1, \qquad Z = \frac{dw}{dt} - u\theta_2 + v\theta_1.$$

500. If the origin of coordinates is also in motion these equations require some slight modification. Let p, q, r be the resolved parts of the velocity of the origin in the directions of the axes. In order that u, v, w may represent the

resolved velocities of the particle P in space (i.e. referred to an origin fixed in space), we must add p, q, r respectively to the expressions given for u, v, w in Art. 499. These additions having been made, u, v, w represent the component space velocities of P, and the expressions for the space accelerations X, Y, Z are the same as those given above. See Art. 227.

The theory of moving axes is more fully given in the author's treatise on *Rigid Dynamics*. The demonstration here given of the fundamental theorem is founded on a method used by Prof. Slesser in the *Quarterly Journal*, 1858. Another simple proof is given in the chapter on moving axes at the beginning of vol. II. of the treatise just referred to.

501. Moving field of force. *When the field of force is fixed relatively to axes moving about a fixed origin we may obtain the equation corresponding to that of vis viva.*

If T be the semi vis viva, we know that dT/dt is equal to the sum of the virtual moments of the forces divided by dt. Hence, the mass being unity,

$$\frac{dT}{dt} = Xu + Yv + Zw$$
$$= Xx' + Yy' + Zz' + \theta_3 (xY - yX) + \ldots\ldots$$

If A_1, A_2, A_3 are the angular momenta about the axes (Art. 492),

$$A_1 = yw - zv, \qquad A_2 = zu - xw, \qquad A_3 = xv - yu,$$

and, taking moments about the axes,

$$dA_1/dt = yZ - zY, \qquad dA_2/dt = zX - xZ, \qquad dA_3/dt = xY - yX.$$

The equation of vis viva therefore becomes

$$\frac{dT}{dt} - \theta_1 \frac{dA_1}{dt} - \theta_2 \frac{dA_2}{dt} - \theta_3 \frac{dA_3}{dt} = \frac{dU}{dt},$$

where U is a function of the coordinates x, y, z only. If θ_1, θ_2, θ_3 are constant, this, when integrated, reduces to the equation of Art. 256.

502. *Ex.* 1. Show how to deduce the polar forms (C), Art. 493, from the equations for moving axes.

Let the moving axes be represented by $O\xi$, $O\eta$, $O\zeta$. Let the axis of ξ move so as always to coincide with the radius vector OP; let $O\eta$ be always perpendicular to the plane zOP. The angular velocity $d\theta/dt$ of the radius vector may therefore be represented by $\theta_2 = d\theta/dt$ about $O\eta$. The plane zOP has an angular velocity $d\phi/dt$ about Oz, and this may be resolved into $\theta_1 = \cos\theta\, d\phi/dt$ and $\theta_3 = \sin\theta\, d\phi/dt$. Also the coordinates of P are $\xi = r$, $\eta = 0$, $\zeta = 0$.

It immediately follows from the equations of moving axes that $u = dr/dt$, $v = \theta_3 r$, $w = -\theta_2 r$. Substituting these in the expressions for X, Y, Z we obtain the components of acceleration already written at length in Art. 493.

Ex. 2. If $(\alpha_1\beta_1\gamma_1)$, $(\alpha_2\beta_2\gamma_2)$, $(\alpha_3\beta_3\gamma_3)$ are the direction cosines of a system of orthogonal axes moving about the origin, prove that

$$\theta_3 = \frac{d\alpha_1}{dt}\alpha_2 + \frac{d\beta_1}{dt}\beta_2 + \frac{d\gamma_1}{dt}\gamma_2,$$

where θ_3 is positive when the rotation is from the first axis to the second.

To prove this we notice that θ_3 measures the rate at which the axis of y is separating from the position of the axis of x at the time t. Hence $-\theta_3 dt$ is the cosine of the angle the new axis of y makes with the old axis of x.

Ex. 3. A particle is describing an orbit about a centre of force which varies as any function of the distance, and is acted on by a disturbing force which is always perpendicular to the plane of the instantaneous orbit and is inversely proportional to the distance of the particle from the centre of force. Prove that the plane of the instantaneous orbit revolves uniformly round its instantaneous axis.

[Math. Tripos, 1860.]

Lagrange's Equations.

503. Lagrange has given a general theorem by which we can form the equations of motion of a particle, or of a system of particles, in any kind of coordinates*.

The expression "coordinate" is here used in a generalized sense. *Any quantities are called the coordinates of a particle, or of a system of particles, which determine the position of that particle or system in space.*

In using Lagrange's equations, it will be found convenient to represent by some special symbols, such as accents, all total differential coefficients with regard to the time; thus x', x'' represent respectively dx/dt and d^2x/dt^2.

504. Lemma. *Let L be a function of any variables x, y, &c., their velocities x', y', &c., and the time t. If we express x, y, &c. as functions of some independent variables θ, ϕ, &c. and the time t, say*

$$x = f(t, \theta, \phi, \&c.), \quad y = F(t, \theta, \phi, \&c.), \quad z = \&c. \ldots\ldots (1),$$

then will

$$\frac{d}{dt}\frac{dL}{d\theta'} - \frac{dL}{d\theta} = \left(\frac{d}{dt}\frac{dL}{dx'} - \frac{dL}{dx}\right)\frac{dx}{d\theta} + \left(\frac{d}{dt}\frac{dL}{dy'} - \frac{dL}{dy}\right)\frac{dy}{d\theta} + \&c.$$

Representing partial differential coefficients by suffixes, we have by differentiating (1),

$$x' = f_t + f_\theta \theta' + f_\phi \phi' + \&c. \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Since θ enters into the expression L through both x, y, &c. and their velocities x', y', &c. while θ' enters only through x', y', &c.,

* The Lagrangian equations are of the greatest importance in the higher dynamics and are usually studied as a part of *Rigid Dynamics*. We give here only such theorems as may be of use in the rest of this treatise. The application of the method to impulses, to the cases in which the geometrical equations contain the differential coefficients of the coordinates, the use of indeterminate multipliers, the Hamiltonian function, &c., are regarded as a part of the higher dynamics.

we have the partial differential coefficients

$$\frac{dL}{d\theta}=\frac{dL}{dx}\frac{dx}{d\theta}+\frac{dL}{dx'}\frac{dx'}{d\theta}+\&c. \qquad\qquad (3),$$

$$\frac{dL}{d\theta'}=\frac{dL}{dx'}\frac{dx'}{d\theta'}+\&c. \qquad\qquad (4),$$

where in each case the &c. represents the corresponding terms for y, z, &c.

By differentiating (2) we see that $\dfrac{dx'}{d\theta'}=f_\theta=\dfrac{dx}{d\theta}$. Hence

$$\frac{d}{dt}\frac{dL}{d\theta'}-\frac{dL}{d\theta}=\left(\frac{d}{dt}\frac{dL}{dx'}-\frac{dL}{dx}\right)\frac{dx}{d\theta}+\&c.$$
$$+\frac{dL}{dx'}\left(\frac{d}{dt}f_\theta-\frac{dx'}{d\theta}\right)+\&c. \qquad\qquad (5).$$

By differentiating f_θ totally with regard to t, we have

$$\frac{d}{dt}f_\theta=f_{\theta t}+f_{\theta\theta}\theta'+\&c. \qquad\qquad (6).$$

The right-hand side of this equation is seen by differentiating (2) to be equal to $\dfrac{dx'}{d\theta}$. It therefore follows that all the terms in the second line of (5) vanish. The lemma has therefore been proved.

505. By using this lemma we may deduce Lagrange's equations from the Cartesian equations of motion. For the sake of generality, let there be any number of particles, of any masses m_1, m_2, &c., and let their coordinates be (x_1, y_1, z_1), (x_2, y_2, z_2), &c. Let T be the semi vis viva of the system, then

$$2T=\Sigma m\,(x'^2+y'^2+z'^2) \qquad\qquad (7).$$

Let U be the work function of the impressed forces, then U is a function of the coordinates only. Let R_x, R_y, R_z be the components of any forces of constraint which act on the typical particle m. We have as many Cartesian equations of motion of the form

$$mx''-\frac{dU}{dx}=R_x, \quad my''-\frac{dU}{dy}=R_y, \quad mz''-\frac{dU}{dz}=R_z,$$

as there are particles.

The particles may be free or connected together, or constrained by curves and surfaces, but after using all the given geometrical relations, the position of the system may be made to depend on some *independent* auxiliary quantities or coordinates. Let these

be θ, ϕ, &c.; then writing $L = T + U$, we have for the particle m,

$$\frac{d}{dt}\frac{dL}{dx'} - \frac{dL}{dx} = \frac{d}{dt}mx' - \frac{dU}{dx} = R_x,$$

with similar forms for y and z. Hence using the lemma,

$$\frac{d}{dt}\frac{dL}{d\theta'} - \frac{dL}{d\theta} = \Sigma\left(R_x\frac{dx}{d\theta} + R_y\frac{dy}{d\theta} + R_z\frac{dz}{d\theta}\right) \quad \text{.........(8)},$$

where Σ implies summation for all the particles.

The right-hand side of this equation (after multiplication by $\delta\theta$) is the virtual moment of the forces of constraint for a geometrical displacement $\delta\theta$. This by the principle of virtual work is known to be zero.

Since the variations of the coordinates x, y, &c. due to the displacement $\delta\theta$ are deduced from the partial differential coefficients $dx/d\theta$, $dy/d\theta$, &c., t not varying, the displacement given to the system is one consistent with the geometrical relations as they exist at the instant of time t.

Taking the various kinds of forces of constraint it has been proved in Art. 248 that the virtual moment of each for such a displacement is zero. Consider the case of a particle constrained to rest on a curve or surface, the virtual moment is zero for any displacement tangential to the *instantaneous* position of the curve or surface. *The restriction that the geometrical equations must not contain the time explicitly is not necessary in Lagrange's equations.*

If some of the particles are connected together so as to form a rigid body, the mutual actions and reactions of the molecules are equal. Their virtual moments destroy each other because each pair of particles remain at a constant distance from each other. The Lagrangian equations may therefore be applied to rigid bodies.

506. *The Lagrangian equations of motion* are therefore

$$\frac{d}{dt}\frac{dL}{d\theta'} - \frac{dL}{d\theta} = 0, \quad \frac{d}{dt}\frac{dL}{d\phi'} - \frac{dL}{d\phi} = 0, \quad \&\text{c.} = 0 \text{ ...... (9)}.$$

The function $L = T + U$ and is therefore *the sum of the kinetic energy and the work function.* If we use the function V to represent the potential energy, we have, by definition, $U + V$ equal to a constant. We then put $L = T - V$, so that L *is the difference between the kinetic and potential energies.* Substituting these values for L, and remembering that U and V are functions of the coordinates and not of their velocities, we may also write the Lagrangian equations in the two typical forms

$$\frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta} = \frac{dU}{d\theta}; \qquad \frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta} + \frac{dV}{d\theta} = 0 \text{......(10)},$$

where θ stands for any one of the coordinates. It should be

noticed that in these equations, all the differential coefficients are partial, except those with regard to t.

The function L is sometimes called *the Lagrangian function*. We see that *when once it has been found, all the dynamical equations, free from all unknown reactions, can be deduced by simple differentiation.*

507. Virtual moment of the effective forces. If we substitute for L in the lemma of Art. 504 the value of T given by (7) we have

$$\frac{d}{dt}\frac{dT}{d\theta'}-\frac{dT}{d\theta}=\Sigma\left\{mx''\frac{dx}{d\theta}+my''\frac{dy}{d\theta}+\&c.\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ (11).$$

The right-hand side (after multiplication by $\delta\theta$) is the sum of the virtual moments of the effective forces mx'', my'', &c. It follows therefore that *the Lagrangian expression on the left-hand side (after multiplication by $\delta\theta$) represents the sum of the virtual moments of the effective forces, when expressed in terms of the generalized coordinates θ, ϕ, &c.*

In the same way writing T for the arbitrary function L in (4), we have by (7)

$$\frac{dT}{d\theta'}=\Sigma\left\{mx'\frac{dx}{d\theta}+my'\frac{dy}{d\theta}+\&c.\right\}.$$

The left-hand side (after multiplication by $\delta\theta$) therefore represents the sum of the virtual moments of the momenta of the several particles of the system for the displacement $\delta\theta$. It is often called *the generalized θ component* of the momentum.

508. Meaning of the lemma. The fundamental equation represented by the lemma has been deduced from the principles of the differential calculus without reference to any mechanical theorem.

Analytically, it expresses the fact that the Lagrangian operator symbolized by

$$\underset{\smile}{\Delta}_\theta=\frac{d}{d\theta}-\frac{d}{dt}\frac{d}{d\theta'}$$

follows the same law as the differential coefficient $d/d\theta$, i.e.

$$\Delta_\theta L=\Delta_x L\,.\,\frac{dx}{d\theta}+\Delta_y L\,.\,\frac{dy}{d\theta}+\ldots\ldots,$$

which may also be written

$$\Delta_\theta L\,.\,\delta\theta=\Delta_x L\,.\,\delta x+\Delta_y L\,.\,\delta y+\ldots\ldots,$$

where $\delta\theta$, δx, δy, &c. are any small arbitrary variations consistent with the geometrical relations which hold at the time t.

If we interpret the lemma *dynamically* (Art. 506), the equation asserts that the sum of the virtual moments of the effective and impressed forces for a displacement $\delta\theta$ has the same value whatever changes are made in the coordinates.

509. Working rule. When we solve a dynamical problem we begin by writing down the equation of vis viva, viz. $T=U+C$.

It appears that when we have done this, Lagrange's method enables us to write down all the equations of motion of the second order by performing certain differentiations on the quantities on each side of the equation (Art. 506).

We shall presently show that before performing these differentiations, we may remove certain factors from one side to the other by making a change in the independent variable t; Art. 524.

510. The function T. We have assumed that the Cartesian coordinates x, y, z of every particle of the system can be expressed in terms of the generalized coordinates θ, ϕ, &c. by means of equations of the form

$$x = f(t, \theta, \phi, \&c.) \qquad\qquad (1);$$

these equations may contain t, but not θ', ϕ', &c. (Art. 504). *In choosing therefore the Lagrangian coordinates, we see that they must be such that the Cartesian coordinates of every particle could be expressed if required in terms of them by means of equations which may contain the time, but do not contain differential coefficients with regard to the time.*

Differentiating the geometrical equations (1) as in Art. 504

$$x' = f_t + f_\theta \theta' + f_\phi \phi' + \&c., \quad y' = \&c. \qquad\qquad (2),$$

and substituting in the expression for the vis viva

$$2T = \Sigma m (x'^2 + y'^2 + z'^2) \qquad\qquad (7),$$

given in Art. 505, we observe that $2T$ takes the form

$$2T = A_{11}\theta'^2 + 2A_{12}\theta'\phi' + \ldots + B_1\theta' + B_2\phi' + \ldots + C,$$

where the coefficients A_{11}, &c., B_1, B_2, &c., and C are functions of t, θ, ϕ, &c.

In most dynamical problems, *the geometrical equations do not contain the time explicitly,* i.e. t does not enter into the equations (1) except implicitly through θ, ϕ, &c. The term f_t will therefore be absent from the equation (2), Art. 504. Hence x', y', z' are homogeneous functions of θ', ϕ', &c. of the first order. When substituted in (7), we find that $2T$ is a *homogeneous function of θ', ϕ', &c. of the second order, viz.*

$$2T = A_{11}\theta'^2 + 2A_{12}\theta'\phi' + \ldots,$$

where A_{11}, A_{12}, &c. are functions of the coordinates θ, ϕ, &c. but not of t.

511. Examples of Lagrange's equations. *Ex.* Two particles, of masses M, m, are connected by a light rod, of length l. The first A is constrained to move along a smooth fixed horizontal wire, while the other B is free to oscillate in the vertical plane under the action of gravity. It is required to find the motion.

To fix the positions of both the particles in space, we require two coordinates, say, the distance ξ of the point A from some origin O and the inclination θ of AB

to the vertical. The Cartesian coordinates of B are then $x=\xi+l\sin\theta$ and $y=l\cos\theta$. The semi vis viva and work functions are then

$$T=\tfrac{1}{2}M\xi'^2+\tfrac{1}{2}m\{(\xi'+l\cos\theta\theta')^2+(l\sin\theta\theta')^2\}$$
$$=\tfrac{1}{2}(M+m)\xi'^2+ml\cos\theta\xi'\theta'+\tfrac{1}{2}ml^2\theta'^2 \quad\ldots\ldots (1),$$
$$U=mgl\cos\theta \quad\ldots\ldots (2).$$

Substituting in the Lagrangian equations,

$$\frac{d}{dt}\frac{dT}{d\xi'}-\frac{dT}{d\xi}=\frac{dU}{d\xi},\qquad \frac{d}{dt}\frac{dT}{d\theta'}-\frac{dT}{d\theta}=\frac{dU}{d\theta},$$

we have

$$\left.\begin{aligned}&\frac{d}{dt}\{(M+m)\xi'+ml\cos\theta\theta'\}=0\\ &\frac{d}{dt}\{ml\cos\theta\xi'+ml^2\theta'\}+ml\sin\theta\xi'\theta'=-mgl\sin\theta\end{aligned}\right\}.$$

These give

$$(M+m)\xi'+ml\cos\theta\theta'=A,\qquad \cos\theta\xi''+l\theta''=-g\sin\theta \quad\ldots\ldots (3),$$

where A is a constant of integration. Eliminating ξ, we have

$$(M+m\sin^2\theta)\theta'\theta''+m\sin\theta\cos\theta\theta'.\theta'^2=-\frac{g}{l}(M+m)\sin\theta\theta'.$$

This gives by integration

$$(M+m\sin^2\theta)\theta'^2=C+\frac{2g}{l}(M+m)\cos\theta \quad\ldots\ldots (4).$$

In this way the velocities ξ' and θ' have been found in terms of the coordinates ξ, θ.

We have here used both the Lagrangian equations, but we might have replaced the second by the equation of vis viva, viz. $T=U+C$. Eliminating ξ' by the help of the first of equations (3), we should then have arrived at the result (4) without any further integrations.

512. *Ex.* 1. *The four elementary forms for the acceleration of a point follow at once from Lagrange's equations.* For example, let us deduce the polar form given in Art. 493.

We notice that the components of velocity of P along the radius vector and perpendicular to it, are respectively r' and $r\theta'$, while that perpendicular to the plane zOP is $r\sin\theta\phi'$. Since these three directions are orthogonal, we have

$$2T=m(r'^2+r^2\theta'^2+r^2\sin^2\theta\phi'^2).$$

Substituting in the Lagrangian equation

$$\frac{d}{dt}\frac{dT}{d\xi'}-\frac{dT}{d\xi}=\frac{dU}{d\xi},$$

where ξ in turn stands for r, θ, ϕ, we obtain

$$\frac{d}{dt}(mr')-m(r\theta'^2+r\sin^2\theta\phi'^2)=\frac{dU}{dr},$$
$$\frac{d}{dt}(mr^2\theta')-mr^2\sin\theta\cos\theta\phi'^2=\frac{dU}{d\theta},$$
$$\frac{d}{dt}(mr^2\sin^2\theta\phi')=\frac{dU}{d\phi},$$

which evidently reduce to the forms given in Art. 493.

Ex. 2. *To deduce the accelerations for moving axes from Lagrange's equations when the component velocities are known.*

We have given by Art. 499,

$$u=x'-y\theta_3+z\theta_2, \qquad v=y'-z\theta_1+x\theta_3, \qquad w=z'-x\theta_2+y\theta_1.$$

Also $$T=\tfrac{1}{2}(u^2+v^2+w^2),$$

the mass of the particle being unity. Since x' enters into the expression for T only through u, while x enters through both v and w, we have

$$\frac{dT}{dx'}=\frac{dT}{du}\frac{du}{dx'}=u, \qquad \frac{dT}{dx}=\frac{dT}{dv}\frac{dv}{dx}+\frac{dT}{dw}\frac{dw}{dx}=v\theta_3-w\theta_2.$$

The Lagrangian equation $$\frac{d}{dt}\frac{dT}{dx'}-\frac{dT}{dx}=\frac{dU}{dx}$$

becomes $$\frac{du}{dt}-v\theta_3+w\theta_2=X.$$

Ex. 3. *To deduce the equation of vis viva from Lagrange's equations.*

Multiplying the Lagrangian equations

$$\frac{d}{dt}\frac{dT}{d\theta'}-\frac{dT}{d\theta}=\frac{dU}{d\theta}, \qquad \frac{d}{dt}\frac{dT}{d\phi'}-\frac{dT}{d\phi}=\frac{dU}{d\phi}, \text{ \&c.,}$$

by θ', ϕ', &c. respectively and adding the results, we have

$$\Sigma\left\{\frac{d}{dt}\left(\theta'\frac{dT}{d\theta'}\right)-\theta''\frac{dT}{d\theta'}\right\}-\Sigma\theta'\frac{dT}{d\theta}=\Sigma\theta'\frac{dU}{d\theta},$$

where Σ implies summation for all the coordinates.

If the geometrical equations do not contain the time explicitly, T is a homogeneous function of θ', ϕ', &c., Art. 510, and by Euler's theorem $\Sigma\theta'\dfrac{dT}{d\theta'}=2T$. Also since T and U are not functions of t,

$$\frac{dT}{dt}=\Sigma\left(\theta'\frac{dT}{d\theta}+\theta''\frac{dT}{d\theta'}\right), \qquad \frac{dU}{dt}=\Sigma\theta'\frac{dU}{d\theta}.$$

Substituting in the expression given above, we have

$$2\frac{dT}{dt}-\frac{dT}{dt}=\frac{dU}{dt}; \qquad \therefore\ T=U+C,$$

where C is an arbitrary constant, usually called the constant of vis viva.

Ex. 4. The position of a moving point is determined by the radii $1/\xi$, $1/\eta$, $1/\zeta$ of the three spheres which pass through it and touch three fixed rectangular coordinate planes at the origin. Find the component velocities u, v, w of the point in the directions of the outward normals of the spheres, and prove that the component accelerations in the same directions are $du/dt+v(\eta u-\xi v)-w(\xi w-\zeta u)$, and two similar expressions. [Coll. Ex. 1896.]

Writing $D=\xi^2+\eta^2+\zeta^2$ we deduce from the equations of the spheres that $x=2\xi/D$, &c. Noticing that the spheres are orthogonal, we find, by resolving the velocities x', y', z' along them, $u=-x\xi'/\xi$, $v=-y\eta'/\eta$, $w=-z\zeta'/\zeta$. Hence

$$T=\tfrac{1}{2}(u^2+v^2+w^2)=2(\xi'^2+\eta'^2+\zeta'^2)/D^2.$$

Also the acceleration along the ξ axis is dU/udt or $-\frac{1}{2}D\,dU/d\xi$. Substituting in the Lagrangian formula $\dfrac{dU}{d\xi}=\dfrac{d}{dt}\dfrac{dT}{d\xi'}-\dfrac{dT}{d\xi}$, we obtain the required result. It may also be deduced from the formulæ of Arts. 499, 502, Ex. 2.

513. *To apply the Lagrangian equations to determine the small oscillations of a system of particles about a position of equilibrium, when the geometrical equations do not contain the time explicitly.*

Let the system have n coordinates and let these be θ, ϕ, &c. Let their values in the position of equilibrium be α, β, &c., and at any time t, let $\theta = \alpha + x$, $\phi = \beta + y$, &c.

The vis viva being a homogeneous function of θ', ϕ', &c. (Art. 510), we have

$$2T = P\theta'^2 + 2Q\theta'\phi' + R\phi'^2 + \&c.,$$

where P, Q, &c. are functions of θ, ϕ, &c. When we substitute $\theta = \alpha + x$, &c. *and reject all powers of the small quantities above the second,* this reduces to an expression of the form

$$2T = A_{11}x'^2 + 2A_{12}x'y' + A_{22}y'^2 + \&c. \ldots\ldots\ldots\ldots(1),$$

where the coefficients are constant, and are known functions of α, β, &c.

The work function U is a function of θ, ϕ, &c. and when expanded takes the form

$$2U = 2U_0 + 2B_1x + 2B_2y + \&c. + B_{11}x^2 + 2B_{12}xy + \&c.\ldots(2).$$

We assume that these expansions are possible.

Since the system is in equilibrium in the position defined by $x = 0$, $y = 0$, &c., we have by the principle of virtual work,

$$\frac{dU}{dx} = 0, \quad \frac{dU}{dy} = 0, \ \&c.; \ \therefore \ B_1 = 0, \ B_2 = 0, \ \&c. \ldots\ldots(3).$$

If the position of equilibrium is not known beforehand, the values of α, β, &c. may be obtained by solving the n equations (3).

To find the equations of motion we substitute in the n Lagrangian equations typified by

$$\frac{d}{dt}\frac{dT}{dx'} - \frac{dT}{dx} = \frac{dU}{dx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Since the expansion for T does not contain the coordinates x, y, &c., we have $dT/dx = 0$, $dT/dy = 0$, &c. The equation (4) therefore becomes

$$\left.\begin{aligned} A_{11}x'' + A_{12}y'' + A_{13}z'' + \&c. &= B_{11}x + B_{12}y + B_{13}z + \&c. \\ A_{12}x'' + A_{22}y'' + A_{23}z'' + \&c. &= B_{12}x + B_{22}y + B_{23}z + \&c. \\ \&c. &= \&c. \end{aligned}\right\} \ldots(5).$$

To solve the equations (5) we follow the rules given in Art. 292. Let any *principal oscillation* be represented by

$$x = G\sin(pt+\alpha), \qquad y = H\sin(pt+\alpha), \text{ \&c.} \quad\text{.........(6)},$$

where G, H, &c. are constants. We find by an easy substitution

$$\left.\begin{aligned}(A_{11}p^2+B_{11})\,G+(A_{12}p^2+B_{12})\,H+\ldots&=0\\(A_{12}p^2+B_{12})\,G+(A_{22}p^2+B_{22})\,H+\ldots&=0\\\text{\&c.}&=0\end{aligned}\right\}\quad\text{.........(7)}.$$

Eliminating the ratios $G : H :$ &c., the n values of p^2 are given by the Lagrangian equation

$$\begin{vmatrix} A_{11}p^2+B_{11}, & A_{12}p^2+B_{12}, & \text{\&c.} \\ A_{12}p^2+B_{12}, & A_{22}p^2+B_{22}, & \text{\&c.} \\ \text{\&c.} & \text{\&c.} & \text{\&c.} \end{vmatrix} = 0\quad\text{...........(8)}.$$

514. It is shown in the higher dynamics that, because the vis viva $2T$ is necessarily positive for all real values of x', y', &c., the values of p^2 given by this determinantal equation are real. If all the roots are positive the values of p are real, and the system of particles then oscillates about the position of equilibrium. If any or all the values of p^2 are negative, some or all the values of p take the form $\pm q\sqrt{-1}$. The corresponding trigonometrical terms in (6) become exponential and the system does not oscillate. See Art. 120.

515. If a value of p^2 is zero p has two *equal* zero values, and the corresponding term in (6) takes the form $A+Bt$. In such a case the coordinate may become large and the system will then depart so far from the position of equilibrium that it will be necessary to take account of the small terms in (1) and (2) of higher orders than the second.

516. Rule. When applying Lagrange's equations to any special case of oscillation *about a position of equilibrium* we begin by writing down the expressions for the vis viva and work function for the system in *its displaced position*, and express these in the *quadratic forms* (1) and (2) (Art. 513). If the whole motion is required we follow in each special case the process described in the general investigation. But if, as usually happens, only the periods are required, we omit the intervening steps and deduce the determinant (8) immediately from the expansions (1), (2).

To help the memory, we notice that, *if we drop the accents in the expression for T, the determinant* (8) *is the discriminant of the quadric $Tp^2 + U$.*

517. *To apply Lagrange's equations to determine the initial motion of a system.*

The method has been already explained in Art. 282. The Lagrangian equations give the values of θ'', ϕ'', &c. in the initial position without introducing the unknown reactions. Differentiating the Lagrangian equations of Art. 506 we obtain θ''', ϕ''', &c., and any higher differential coefficients.

If x, y, z are the Cartesian coordinates of any point P of the system, we have by Art. 510,

$$x = f_1(\theta, \phi, \&c.), \quad y = f_2(\theta, \phi, \&c.), \quad z = \&c.,$$

and therefore by differentiation the initial values of x', x'', &c., y', y'', &c., z', &c. may be found. The initial radius of curvature follows from the formulæ of the differential calculus, Art. 280.

518. *Let, for example, the initial accelerations be required when the system starts from rest.* The initial position being $\theta = \alpha$, $\phi = \beta$, &c. we put, as in Art. 513, $\theta = \alpha + x$, $\phi = \beta + y$, &c. Since the system starts from rest, the velocities x', y', &c. are small and we can make the expansions (1) and (2) as before. Since the initial position is not one of equilibrium, we no longer have $B_1 = 0$, $B_2 = 0$, &c. Retaining only the lowest powers of x, y, &c. which occur in the equations of motion, we have

$$\left.\begin{aligned} A_{11}x'' + A_{12}y'' + \&c. &= B_1 \\ A_{12}x'' + A_{22}y'' + \&c. &= B_2 \\ \&c. &= \&c. \end{aligned}\right\}$$

These determine the initial accelerations of the coordinates and therefore the component accelerations of every point of the system.

519. *Ex.* 1. Let us apply the Lagrangian equations to find the small oscillations of the two particles described in Art. 511.

The quantities ξ, θ represent the deviations of the rod from its position of equilibrium. The vis viva and work function expressed in quadratic forms are

$$T = \tfrac{1}{2}(M+m)\xi'^2 + ml\xi'\theta' + \tfrac{1}{2}ml^2\theta'^2, \qquad U = mgl(1 - \tfrac{1}{2}\theta^2).$$

The determinant is the discriminant of

$$Tp^2 + U = \tfrac{1}{2}(M+m)p^2\xi^2 + mlp^2\xi\theta + \tfrac{1}{2}ml(lp^2 - g)\theta^2;$$

$$\therefore \begin{vmatrix} (M+m)p^2, & mlp^2 \\ mlp^2, & ml(lp^2 - g) \end{vmatrix} = 0.$$

One principal motion is given by

$$p^2 = \frac{g}{l}\frac{M+m}{M}, \qquad \xi = G\sin(pt+\alpha), \qquad \theta = H\sin(pt+\alpha).$$

The other is determined by $p^2=0$; this implies that one coordinate takes the form $A+Bt$. It is evident that the rod could be so projected along the horizontal wire that ξ has this form while $\theta=0$.

The student should apply Lagrange's equations to the problems on small oscillations and initial motions already considered in the chapter on motion in two dimensions. He will thus be able to form a comparison of the advantages of the different methods.

Ex. 2. Three uniform rods AB, BC, CD have lengths $2a$, $2b$, $2a$ and masses m, m', m. They are hinged together at B and C, and at A, D are small smooth rings which are free to move along a fixed fine horizontal bar. The rods hang in equilibrium, forming with the bar a vertical rectangle. When a slight symmetrical displacement is given, the period of a small oscillation is given by $4map^2 = 3g(m+m')$. Find also the periods when the displacement is unsymmetrical. [Coll. Ex. 1897.]

Ex. 3. Two equal strings AC, BC have their ends at the fixed points A, B, on the same horizontal line, and at C a heavy particle is attached. From C a string CD hangs down with a second heavy particle at D. Find the periods of the three small oscillations. [The two periods of the oscillations perpendicular to the vertical plane through A and B are given in Art. 300, Ex. 1.]

520. Solution of Lagrange's Equations. Our success in obtaining the first integrals of the Lagrangian equations will greatly depend on the choice of coordinates. When the position of the system is determined by only one coordinate, the equation of vis viva is the first integral, and this is sufficient to determine the motion.

When there are two or more coordinates, integrals can be found only in special cases. The general problem of the solution of the Lagrangian equations is too great a subject to be attempted here. It is sufficient to state a few elementary rules which may assist the student.

521. We should, if possible, *so choose the coordinates that some one of them is absent from the expression for the work function U.* For example, if there be any direction such that the component of the impressed forces is zero throughout the motion, we should take the axis of z in that direction and let z be one of the coordinates. Again if the moment of the forces about some straight line fixed in space, say Oz, is always zero, the angle ϕ which the plane POz makes with xOz will be a suitable coordinate. In that case $dU/d\phi = 0$ and U is independent of ϕ. These, or similar,

mechanical considerations generally enable us to make a proper choice.

Let θ be the coordinate absent from the work function, then *if θ is also absent from the expression for T*, though the differential coefficient θ' is present, the Lagrangian equation

$$\frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta} = \frac{dU}{d\theta} \text{ becomes } \frac{dT}{d\theta'} = A,$$

where A is the constant of integration. Thus a first integral, *different from that of vis viva,* has been found.

522. Liouville's integral. Liouville has given an integral of Lagrange's equations which has the advantage of great simplicity when it can be applied. This may be found in vol. XI. of his *Journal*, 1846; the following is a slight modification of his method.

Let us suppose that the vis viva has the form

$$2T = M(P\theta'^2 + Q\phi'^2 + R\psi'^2 + \&c.) \qquad (1),$$

where the products $\theta'\phi'$, $\phi'\psi'$, &c. are absent. *The method requires that the coefficient P should be a function of θ only, while Q, R, &c., are not functions of θ.* We notice that M may be a function of all or any of the coordinates, and Q, R, &c. functions of any except θ. *It is also necessary that* the impressed forces should be such that *the work function U has the form*

$$M(U + C) = F_1(\theta) + F(\phi, \psi, \&c.) \qquad (2),$$

where C is the constant in the equation of vis viva,

$$T = U + C \qquad (3).$$

We shall now prove that *when these conditions are satisfied, a first integral is*

$$\tfrac{1}{2}M^2P\theta'^2 = F_1(\theta) + A \qquad (4).$$

We first put $P\theta'^2 = \xi'^2$, then ξ is a function of θ only and we may temporarily take ξ, ϕ, ψ, &c. as the coordinates. We now have

$$T = \tfrac{1}{2}M(\xi'^2 + Q\phi'^2 + \&c.) = U + C,$$

and the Lagrangian equation for ξ is

$$\frac{d}{dt}(M\xi') - \frac{1}{2}\frac{dM}{d\xi}(\xi'^2 + Q\phi'^2 + \&c.) = \frac{dU}{d\xi}.$$

Using the equation of vis viva, this takes the form

$$M\frac{d}{dt}(M\xi') = (U + C)\frac{dM}{d\xi} + M\frac{dU}{d\xi} = \frac{d}{d\xi}M(U + C).$$

Substituting on the right-hand side from (2) and multiplying by ξ', we have

$$M\xi'\frac{d}{dt}(M\xi') = \frac{dF_1(\theta)}{d\xi}\xi'.$$

Since $F_1(\theta)$ is a function of ξ and not of any of the other coordinates, this gives by an easy integration

$$\tfrac{1}{2}M^2\xi'^2 = F_1(\theta) + A.$$

Returning to the coordinate θ, we have the integral (4).

When the initial conditions are given, the value of C can be found by introducing these conditions into the equation of vis viva. If a solution is required for all

initial conditions C is arbitrary and in that case the condition (2) requires that both MU and M should have the general form indicated on the right-hand of that equation. If

$$M(U+C)=F_1(\theta)+F_2(\phi)+\&c.,$$

and Q, R, &c. are respectively functions of ϕ, ψ, &c. only, it is evident that the method supplies *all* the first integrals.

Ex. If $T=M(P\theta'^2+Q\phi'^2)$, $M=f_1(\theta)+f_2(\phi)$, $MU=F_1(\theta)+F_2(\phi)$, *integrate the Lagrangian equations by Liouville's method.* The integrals are

$$\tfrac{1}{2}M^2P\theta'^2=F_1(\theta)+Cf_1(\theta)+A_1, \qquad \tfrac{1}{2}M^2Q\phi'^2=F_2(\phi)+Cf_2(\phi)+A_2,$$

adding these and using the equation of vis viva we see that $A_1+A_2=0$. The paths are then given by

$$\frac{\sqrt{P}\,d\theta}{\sqrt{(F_1+Cf_1+A_1)}}=\frac{\sqrt{Q}\,d\phi}{\sqrt{(F_2+Cf_2+A_2)}}=\frac{\sqrt{2}\,dt}{M}.$$

Multiplying these by f_1, f_2 and adding, the time is found by

$$\frac{f_1\sqrt{P}\,d\theta}{\sqrt{(F_1+Cf_1+A_1)}}+\frac{f_2\sqrt{Q}\,d\phi}{\sqrt{(F_2+Cf_2+A_2)}}=\sqrt{2}\,dt,$$

where all the variables have been separated.

523. Jacobi's integral. If T be a homogeneous function of the coordinates θ, ϕ, &c. of n dimensions and U a homogeneous function of the same coordinates of $-(n+2)$ dimensions, then one integral is

$$\theta\frac{dT}{d\theta'}+\phi\frac{dT}{d\phi'}+\&c.=(n+2)\,Ct+A,$$

where C is the constant of vis viva and A an arbitrary constant.

To prove this, we multiply the Lagrangian equations by θ, ϕ, &c. and add the products. Remembering Euler's theorem on homogeneous functions, we have

$$\theta\frac{d}{dt}\frac{dT}{d\theta'}+\&c.=nT-(n+2)\,U.$$

The left-hand side is the same as

$$\frac{d}{dt}\left(\theta\frac{dT}{d\theta'}+\&c.\right)-\left\{\theta'\frac{dT}{d\theta'}+\&c.\right\}=\frac{d}{dt}\left(\theta\frac{dT}{d\theta'}+\&c.\right)-2T,$$

since T is a homogeneous function of θ', ϕ', &c. of two dimensions. Remembering that $T-U=C$, we have $\dfrac{d}{dt}\left\{\theta\dfrac{dT}{d\theta'}+\&c.\right\}=(n+2)\,C$.

Ex. A free system of particles moves under the influence of their mutual attractions, the law of force being the inverse cube: show that $\Sigma mr^2=A+Bt+Ct^2$ where r is the distance of the particle m from the origin.

[*Vorlesungen über Dynamik.*]

Some developments of these results are given in the first volume of the author's treatise on *Rigid Dynamics.*

524. Change of the independent variable. *It is sometimes useful to be able to change the independent variable in Lagrange's equations from t to some other quantity τ so that $d\tau=P\,dt$, where P is any function of the coordinates.*

We suppose that the geometrical equations do not contain the time explicitly, so that T is a homogeneous function of the form

$$T=\tfrac{1}{2}A_{11}\theta'^2+A_{12}\theta'\phi'+\tfrac{1}{2}A_{22}\phi'^2+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Let suffixes applied to the coordinates mean differentiations with regard to τ just as accents denote differentiations with regard to t. Then

$$\theta' = P\theta_1, \quad \phi' = P\phi_1, \quad \&c.$$

Consider how any one of Lagrange's equations, say,

$$\frac{d}{dt}\frac{dT}{d\phi'} - \frac{dT}{d\phi} = \frac{dU}{d\phi} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

is affected by the change of t. Let us write

$$T_2 = (\tfrac{1}{2}A_{11}\theta_1^2 + A_{12}\theta_1\phi_1 + \ldots)P \quad \ldots\ldots\ldots\ldots (3).$$

Supposing that P is a function of the coordinates only, not of θ', ϕ', &c., we have

$$\frac{dT}{d\phi'} = A_{12}\theta' + A_{22}\phi' + \ldots = (A_{12}\theta_1 + A_{22}\phi_1 + \ldots)P = \frac{dT_2}{d\phi_1},$$

$$\frac{dT}{d\phi} = \frac{1}{2}\frac{dA_{11}}{d\phi}\theta'^2 + \ldots = \left(\frac{1}{2}\frac{dA_{11}}{d\phi}\theta_1^2 + \ldots\right)P^2 = P^2\frac{d}{d\phi}\left(\frac{T_2}{P}\right).$$

The Lagrangian equation therefore becomes after a slight reduction

$$\frac{d}{d\tau}\frac{dT_2}{d\phi_1} - \frac{dT_2}{d\phi} = -\frac{T_2}{P}\frac{dP}{d\phi} + \frac{1}{P}\frac{dU}{d\phi} \quad \ldots\ldots\ldots\ldots (4).$$

If we use the equation of vis viva, viz. $T = U + C$, and notice that $T = PT_2$, the right-hand side of this equation becomes $\frac{d}{d\phi}\frac{U+C}{P}$. The typical Lagrangian form therefore takes the form

$$\frac{d}{d\tau}\frac{dT_2}{d\phi_1} - \frac{dT_2}{d\phi} = \frac{d}{d\phi}\frac{U+C}{P} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

We notice that though $T = PT_2$, they are differently expressed. To obtain the partial differential coefficients of T_2, the quantities θ', ϕ', &c. must be replaced by $P\theta_1$, $P\phi_1$, &c. before differentiation.

Suppose for example that the equation of vis viva (Art. 509) is

$$T = M\{\tfrac{1}{2}A\theta'^2 + \&c.\} = U + C,$$

and that we wish to remove the factor M before deducing the Lagrangian equations. Changing the independent variable so that $d\tau = Pdt$, we deduce the Lagrangian equations by operating on

$$T_2 = MP\{\tfrac{1}{2}A\theta_1^2 + \&c.\}, \qquad U_2 = \frac{U+C}{P}.$$

Choosing $MP = 1$, we have

$$T_2 = \tfrac{1}{2}A\theta_1^2 + \&c., \qquad U_2 = M(U+C).$$

The factor M has thus been transferred from the expression for the vis viva to the work function. Here M is a function of the coordinates only.

We may now change the suffixes into accents if we remember that the differentiations are to be taken with regard to τ instead of t. This difference is of no importance if we require only the paths of the particles and not their positions at any time. If the time also be required, we add the equation $dt = Md\tau$.

525. Orthogonal Coordinates. The Lagrangian equations are much simplified when the expression for T can be put into the form

$$T = \tfrac{1}{2}(P\theta'^2 + Q\phi'^2 + \&c.),$$

where the products $\theta'\phi'$, &c. are absent. We shall now prove that this will be the case when the coordinates of the particle are the parameters of systems of curves or surfaces at right angles.

Let the equations of three systems of surfaces which intersect at right angles be $f_1(x, y, z)=\rho_1$, $f_2(x, y, z)=\rho_2$, $f_3(x, y, z)=\rho_3$ .. . (1), where ρ_1, ρ_2, ρ_3 are three constants or parameters whose values determine which surface of each system is taken. These parameters may be regarded as the coordinates of the point of intersection of the three surfaces.

Such coordinates are called sometimes *orthogonal coordinates* and sometimes *curvilinear coordinates*. Their theory was given by Lamé in his *Leçons sur les coordonnées curvilignes*, 1859. In what follows we adopt his notation as far as possible.

As an example of orthogonal coordinates we call to mind a *system of confocal ellipsoids and hyperboloids of one and two sheets*, the lengths of the major axes being usually taken as the parameters. These are called elliptic coordinates. We may also notice that *all the coordinates in common use, whether Cartesian, cylindrical or polar, are orthogonal*. In the first the point is defined as the intersection of three orthogonal planes, in the second we use a cylinder cut by two planes, and in the third a sphere cut by a right cone and a plane.

Let (a_1, b_1, c_1) be the direction cosines of a normal to the surface whose parameter is ρ_1, then

$$a_1=\frac{1}{h_1}\frac{d\rho_1}{dx}, \qquad b_1=\frac{1}{h_1}\frac{d\rho_1}{dy}, \qquad c_1=\frac{1}{h_1}\frac{d\rho_1}{dz} \quad\text{.....................}(2),$$

where

$$h_1^2=\left(\frac{d\rho_1}{dx}\right)^2+\left(\frac{d\rho_1}{dy}\right)^2+\left(\frac{d\rho_1}{dz}\right)^2 \quad\text{.........................}(3).$$

Let ds_1 be an elementary arc of the intersection of the two surfaces ρ_2, ρ_3; then ds_1 is also an elementary length measured along the normal to the surface ρ_1. As we travel along this arc x, y, z and ρ_1 vary, while ρ_2, ρ_3 are constant. Hence

$$\frac{d\rho_1}{dx}dx+\frac{d\rho_1}{dy}dy+\frac{d\rho_1}{dz}dz=d\rho_1;$$

$$\therefore\ a_1dx+b_1dy+c_1dz=d\rho_1/h_1 \quad\text{..........................}(4).$$

But the left side is the sum of the projections of dx, dy, dz on the normal and is therefore ds_1; hence $ds_1=d\rho_1/h_1$. It follows that the component v_1 of velocity along the normal to the surface ρ_1 is $v_1=\frac{1}{h_1}\frac{d\rho_1}{dt}$. In the same way the components of velocity normal to the other two surfaces may be found, and since these are at right angles,

$$v^2=\frac{1}{h_1^2}\rho_1'^2+\frac{1}{h_2^2}\rho_2'^2+\frac{1}{h_3^2}\rho_3'^2 \quad\text{..........................}(5),$$

where accents denote differential coefficients.

In order to use this expression, it will be necessary to express h_1, h_2, h_3, in terms of the new coordinates ρ_1, ρ_2, ρ_3. To effect this we solve the equations (1) and determine x, y, z as functions of ρ_1, ρ_2, ρ_3; finally substituting these values in the expressions (3) for h_1, h_2, h_3. This is sometimes a lengthy process.

Motion on a Curve.

526. Fixed Curves. *To find the motion of a particle on a smooth curve fixed in space.*

To find the velocity, we resolve the forces along the tangent to the curve. If F be the component of the impressed forces

X, Y, Z, this gives as in Art. 181,

$$mv\frac{dv}{ds} = F = X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}.$$

If U be the work function, $F = dU/ds$, and we have

$$\tfrac{1}{2}mv^2 = U + C,$$

which is the equation of vis viva.

To find the pressure, we resolve in any two directions which may suit the problem under consideration. Supposing that we choose the radius of curvature and binormal, we have

$$\frac{mv^2}{\rho} = G + R_1, \qquad 0 = H + R_2,$$

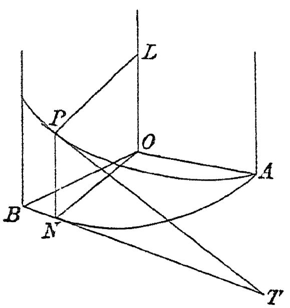
where G, H are the components of the impressed forces; R_1, R_2 the corresponding components of the pressure on the particle.

These equations show that the pressure of the particle on the curve is the resultant of two forces, (1) the statical pressure due to the forces urging the particle against the curve, (2) the centrifugal force mv^2/ρ acting in the direction opposite to that in which ρ is measured, Art. 183.

527. *Ex.* 1. A plane is drawn through the tangent at P making an angle i with the osculating plane. If ρ' be the radius of the circle of closest contact to the curve in this plane, then $\frac{mv^2}{\rho'} = G' + R'$ where G' and R' are the components of the impressed accelerating force and of the pressure respectively.

This follows from the theorem on curves $\rho' \cos i = \rho$, corresponding to Meunier's theorem on surfaces.

Ex. 2. A helix is placed with its axis vertical, and a bead slides on it under the action of gravity. Find the motion and pressure.



Let a be the radius of the cylinder, α the inclination of the tangent to the horizon. Drawing PL perpendicular to the axis of z, the radius of curvature is a length measured along PL equal to $a \sec^2 \alpha$. If PT is the tangent, the osculating plane is LPT. If the helix is smooth we have

$$v^2 = -2gz + C, \qquad \frac{v^2 \cos^2 \alpha}{a} = \frac{R_1}{m},$$

$$0 = g \cos \alpha + \frac{R_2}{m}.$$

If the particle start from rest at a height h, we see that $C = 2gh$. Since $v = -ds/dt$ and $ds \sin \alpha = dz$, we find that the time of descending that height is $\operatorname{cosec} \alpha \sqrt{2h/g}$.

If the helix is rough, the friction is $\mu\sqrt{(R_1^2+R_2^2)}$. Supposing that the coefficient of friction is $\mu=\tan\alpha$, the resolution along the tangent becomes

$$v\frac{dv}{ds}=-g\sin\alpha+\frac{\sin\alpha}{a}\sqrt{(v^4\cos^2\alpha+a^2g^2)},$$

writing $v^2\cos\alpha=\xi$ for brevity, we find

$$\int\frac{d\xi}{\sqrt{(\xi^2+a^2g^2)}-ag}=\frac{s\sin 2\alpha}{a}+C$$

To integrate this we multiply the numerator and denominator of the fraction on the left-hand side by the denominator with the minus sign changed. We then find

$$\log\{v^2\cos\alpha+\sqrt{(v^4\cos^2\alpha+a^2g^2)}\}-\frac{ag+\sqrt{(v^4\cos^2\alpha+a^2g^2)}}{v^2\cos\alpha}=\frac{s\sin 2\alpha}{a}+C.$$

To find C we require the initial value of v. If this were zero the particle would remain at rest because $\mu=\tan\alpha$.

Ex. 3. A rough helical tube of pitch α and radius a is placed so as to have its axis vertical and the coefficient of friction is $\tan\alpha\cos\epsilon$. An extended flexible string which just fits the tube is placed in it: show that when the string has fallen through a vertical distance ma its velocity is $(ag\sec\alpha\sinh 2\mu)^{\frac{1}{2}}$, where μ is determined by the equation

$$\cot\tfrac{1}{2}\epsilon\tanh\mu=\tanh(\mu\sin\epsilon+\tfrac{1}{2}m\cos\alpha\sin 2\epsilon).$$ [Math. Tripos, 1886.]

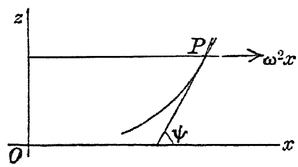
Ex. 4. Two small rings of masses m, m' can slide freely on two wires each of which is a helix of pitch p, the axes being coincident and the principal normals common; the rings repel one another with a force equal to $\mu mm'r$ when they are at a distance r from one another. Prove that if ϕ be the angle the plane through one ring and the axis makes with the plane through the other ring and the axis, the time in which ϕ increases from α to β is $\int_\alpha^\beta\{A\phi^2-2B\cos\phi+C\}^{-\frac{1}{2}}d\phi$, where

$$A=\mu mm'p^2\left\{\frac{1}{m(a^2+p^2)}+\frac{1}{m'(b^2+p^2)}\right\},\qquad B=\frac{ab}{p^2}A,$$

and a, b are the radii of the cylinders on which the helices are drawn.

[Coll. Ex. 1896.]

528. Moving curves. *Ex.* 1. *A particle P is constrained to move on the plane curve $z=f(x)$, which rotates about a straight line Oz in its plane with an angular velocity ω. It is required to form the equations of motion.*



Applying to P an acceleration ω^2x tending from the axis of rotation, we treat the curve as if it were fixed, Art. 495. Taking the tangential and normal resolutions, we have

$$v\frac{dv}{ds}=\frac{F}{m}+\omega^2x\cos\psi,\qquad \frac{v^2}{\rho}=\frac{G}{m}-\omega^2x\sin\psi+\frac{R}{m},$$

where v is the velocity of the particle *relatively to the curve*, ψ the angle the tangent at P makes with the axis of x, and ρ is the radius of curvature. Also F and G are the components of the impressed forces along the tangent and radius of curvature at P.

We may replace the first of these equations by the integral of vis viva, viz.

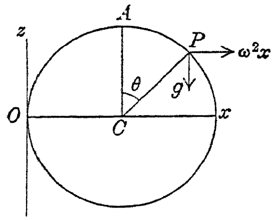
$$\tfrac{1}{2}mv^2=\int(Fds+m\omega^2x\,dx).$$

The second equation then gives the component R of pressure in the plane of the curve. The component R' of pressure perpendicular to the plane of the curve is given by

$$m\frac{1}{x}\frac{d}{dt}(x^2\omega)=H+R',$$

where H is the corresponding component of the impressed force, and x is the distance of the particle from the axis of rotation.

Ex. 2. A circular wire is constrained to turn round a vertical tangent Oz with a uniform angular velocity ω. A heavy smooth bead, starting from the highest point A without any velocity relative to the curve, descends under the action of gravity. Find the velocity and pressure.



Let C be the centre, $OC=a$; let P be the particle, the angle $ACP=\theta$, $v=a\,d\theta/dt$. We reduce the plane to rest by applying to P an accelerating force measured by $\omega^2 x$, where $x=a+a\sin\theta$, and acting parallel to OC. The equation of vis viva then gives

$$\tfrac{1}{2}v^2=g(a-a\cos\theta)+\omega^2\int_a^x x\,dx;$$

$$\therefore\ v^2=2g(a-a\cos\theta)+\omega^2a^2(2\sin\theta+\sin^2\theta).$$

The components R, R' of the pressure on the particle respectively along PC and perpendicular to the plane are given by

$$\frac{v^2}{a}=g\cos\theta-\omega^2x\sin\theta+\frac{R}{m},\qquad \frac{1}{x}\frac{d}{dt}(x^2\omega)=\frac{R'}{m}.$$

The latter equation reduces to $R'=2m\omega v\cos\theta$.

Ex. 3. Two small rings of masses m, m', $(m>m')$ are capable of sliding on a smooth circular wire of radius a, whose vertical diameter is fixed, the rings being below the centre and connected by a light string of length $a\sqrt{2}$: prove that if the wire is made to rotate round the vertical diameter with an angular velocity $\left\{\frac{2g}{a\sqrt{3}}\,\frac{m\sqrt{3}-m'}{m-m'}\right\}^{\frac{1}{2}}$, the rings can be in relative equilibrium on opposite sides of the vertical diameter, the radius through the ring m being inclined at an angle 60° to the vertical. Show also that the tension of the string is $\frac{mm'}{m-m'}\,\frac{\sqrt{3}-1}{\sqrt{2}}\,g$.

[Coll. Ex. 1897.]

Ex. 4. A smooth circular cone of angle 2α has its axis vertical and its vertex, pierced with a small hole, downwards. A mass M hangs at rest by a string which passes through the vertex and a mass m attached to the upper extremity describes a horizontal circle on the inner surface of the cone. Find the time T of a complete revolution, and prove that small oscillations about the steady motion take place in the time $T\operatorname{cosec}\alpha\,\{(M+m)/3m\}^{\frac{1}{2}}$. [Coll. Ex. 1896.]

Ex. 5. A smooth plane revolves with uniform angular velocity ω about a fixed vertical axis which intersects it in the point O, at which a heavy particle is placed at rest. Show that during the subsequent motion $v^2=p^2\omega^2+2gz$; where z is the depth of the particle below O, p its distance from the axis and v the speed with which the path is traced on the plane. [Coll. Ex. 1893.]

529. A case of free motion with two centres of force. *Ex.* 1. *A particle P, of unit mass, is constrained to move along an elliptic wire without inertia which can turn freely about its major axis. The particle is acted on by two centres of force, situated in the foci S, H, which attract according to the law of the inverse square. Prove that the pressure on the curve is zero in certain cases.*

We take the major axis as the axis of z and the origin at the centre. Let ω be the angular velocity of the wire. Representing the distance of the particle P from the major axis by y, the component R' of pressure on the particle perpendicular to the plane of the curve is given by

$$\frac{1}{y}\frac{d}{dt}(y^2\omega)=R'.$$

But since the wire is without inertia, i.e. without mass, the wire moves so that the pressure R' on it is zero, Art. 267. We therefore have throughout the motion

$$y^2\omega=B,$$

where B is the constant of angular momentum about the axis of rotation.

Let the distances of the particle from the foci S, H be r_1, r_2; and let the central forces be μ_1/r_1^2, μ_2/r_2^2.

To find the motion in the plane zOP, we apply to P an acceleration $\omega^2y=B^2/y^3$, tending *from* the major axis, and then treat the curve as if it were fixed. We notice that the particle could freely describe the ellipse under any one of the forces μ_1/r_1^2, μ_2/r_2^2, B^2/y^3 if properly projected; see Arts. 333, 323. It immediately follows that if all the three forces act simultaneously, the pressure on the particle will be a constant multiple of the curvature, Art. 272.

The pressure will be zero, if the square of the velocity of projection is equal to the sum of the squares of the velocities when the particle describes the curve freely under each force separately; Art. 273. We find therefore that if v_1 be the velocity relatively to the curve, the pressure is zero, if

$$v_1^2=\mu_1\left(\frac{2}{r_1}-\frac{1}{a}\right)+\mu_2\left(\frac{2}{r_2}-\frac{1}{a}\right)-B^2\left(\frac{1}{y^2}+\frac{a^2-b^2}{b^4}\right).$$

If v be the resultant velocity of the particle in space, we have $v^2=v_1^2+\omega^2y^2$. Hence

$$v^2=\mu_1\left(\frac{2}{r_1}-\frac{1}{a}\right)+\mu_2\left(\frac{2}{r_2}-\frac{1}{a}\right)-B^2\frac{a^2-b^2}{b^4}.$$

When the pressure is zero, the wire may be removed and the particle describes its path freely in space under the action of the two given centres of force. The general path under all circumstances of projection has not been found. *If the particle is projected along the tangent to any ellipse having S, H for foci with a velocity whose component in the plane of the ellipse is* v_1, *and whose component perpendicular to the plane is* $v'=\omega y=B/y$, *it will continue to describe the ellipse freely, while the ellipse itself moves round the straight line SH with a variable angular velocity* $\omega=B/y^2$.

Ex. 2. If the particle is also acted on by a third centre of force situated at the centre and attracting according to the direct distance, prove that the pressure on the revolving wire is zero in certain cases.

530. *Ex. A particle P of unit mass moves on a smooth curve which is constrained to turn about a fixed axis with an angular velocity ω. It is required to find the relative motion.*

Let the axis of rotation be the axis of z and let the axes of x, y be fixed to the curve and rotate round Oz with the angular velocity ω. Let us refer the motion to these moving axes. Since $\theta_1=0$, $\theta_2=0$, $\theta_3=\omega$, the equations of Art. 499 become

$$\left.\begin{aligned} X+R_1&=\frac{du}{dt}-v\omega, & u&=\frac{dx}{dt}-y\omega\\ Y+R_2&=\frac{dv}{dt}+u\omega, & v&=\frac{dy}{dt}+x\omega\\ Z+R_3&=\frac{dw}{dt}, & w&=\frac{dz}{dt}\end{aligned}\right\}\quad\ldots\ldots\ldots(1),$$

where R_1, R_2, R_3 are the components of the pressure on the particle. Eliminating u, v, w,

$$\left.\begin{aligned} \frac{d^2x}{dt^2}&=X+R_1+\omega^2x+\frac{d\omega}{dt}y+2\omega\frac{dy}{dt}\\ \frac{d^2y}{dt^2}&=Y+R_2+\omega^2y-\frac{d\omega}{dt}x-2\omega\frac{dx}{dt}\\ \frac{d^2z}{dt^2}&=Z+R_3\end{aligned}\right\}\quad\ldots\ldots\ldots(2).$$

The resultant of the two accelerating forces $X_1=\omega^2x$, $Y_1=\omega^2y$ is a force tending directly from the axis of rotation and whose magnitude is $F_1=\omega^2r$, where r is the distance of the particle P from the axis.

The resultant F_2 of the two forces $X_2=y\,d\omega/dt$, $Y_2=-x\,d\omega/dt$ is $F_2=-r\,d\omega/dt$, and it acts perpendicularly to the plane containing the axis of rotation and the particle in the direction in which the angular velocity ω is measured.

To find the resultant F_3 of the forces $X_3=2\omega dy/dt$, $Y_3=-2\omega dx/dt$, we notice that the component along the tangent to the curve, viz. $X_3dx/ds+Y_3dy/ds$, is zero. The resultant acts perpendicularly to the given curve, and may be compounded with and included in the reaction. When only the motion of the particle is required, the force F_3 may be omitted.

Reasoning as in Art. 197, we see that the equations of motion (2) become the same as if the particle were moving on a fixed curve, provided we impress on the particle (in addition to given forces X, Y, Z) *two* accelerating forces, viz. (1) a force $F_1=\omega^2r$ and (2) a force $F_2=-r\,d\omega/dt$.

The process of including the two forces F_1, F_2 among the impressed forces is sometimes called *reducing the curve to rest.*

The curve having been reduced to rest, the velocity of the particle relatively to the curve is found either by the equation of vis viva or by resolving along the tangent. We find

$$\tfrac{1}{2}v^2=C+U+\int\omega^2r\,dr-\int r\frac{d\omega}{dt}\cdot rd\phi,$$

where U represents the work function. If the angular velocity is uniform, this reduces to

$$\tfrac{1}{2}v^2=C+U+\tfrac{1}{2}\omega^2r^2.$$

The velocity thus found is the velocity relative to the curve. The actual velocity in space is the resultant of velocity v and the velocity ωr of the point of the curve instantaneously occupied by the particle.

531. The pressure of the fixed curve on the particle is not the same as the actual pressure of the moving curve. Representing the first by R' and the second by R, we see that R' is the resultant of R and the two forces $X_3 = 2\omega dy/dt$, $Y_3 = -2\omega dx/dt$. We may compound these two forces into a single force F_3. We project the moving curve on a plane perpendicular to the axis of rotation. If P' be the projection of P, dx/dt and dy/dt are the component velocities of P. The resultant is then evidently $F_3 = 2\omega v'$ where v' is the velocity of P' relatively both to the curve and its projection. The direction of F_3 is perpendicular not only to the given curve but also to its projection. The components along and perpendicular to the radius vector are $+2\omega r d\theta/dt$ and $-2\omega dr/dt$.

532. *Ex.* A small bead slides on a smooth circular ring of radius a which is made to revolve about a vertical axis passing through its centre with uniform angular velocity ω, the plane of the ring being inclined at a constant angle α to the horizontal plane. Show that the law of angular motion of the bead on the ring is the same as that of a bead on the ring of radius $a/\sin\alpha$ revolving round a vertical diameter with angular velocity $\omega \sin\alpha$. [Coll. Ex.]

533. A changing curve. *A bead of unit mass moves on a smooth curve whose form is changing in any given manner. It is required to find the motion.*

Let the equations of the curve be written in the form

$$x = f_1(\theta, t), \qquad y = f_2(\theta, t), \qquad z = f_3(\theta, t) \ldots\ldots\ldots \quad (1),$$

where θ is an auxiliary variable. We may regard the position of the particle at any given time t as defined by some value of θ. Our object is to find θ in terms of the time.

Let us use Lagrange's equations. We have

$$T = \tfrac{1}{2}\Sigma\,(f_\theta \theta' + f_t)^2 \ldots\ldots\ldots \quad (2),$$

where Σ implies summation for all the coordinates, and partial differential coefficients are indicated by suffixes. The Lagrangian equation is

$$\frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta} = \frac{dU}{d\theta} \ldots\ldots\ldots \quad (3);$$

$$\therefore\ \Sigma \frac{d}{dt}(f_\theta\theta' + f_t) f_\theta - \Sigma\,(f_\theta\theta' + f_t)\,(f_{\theta\theta}\theta' + f_{\theta t}) = \frac{dU}{d\theta} \ldots\ldots\ldots \quad (4).$$

This is a differential equation of the second order from which θ may be found.

The three components of the pressure on the particle in the directions of the axes may be found by differentiating the equations (1). If X, Y, Z, be the components of the impressed forces; R_1, R_2, R_3 those of the pressure, the Cartesian equations of motion are

$$\frac{d^2x}{dt^2} = X + R_1, \qquad \frac{d^2y}{dt^2} = Y + R_2, \qquad \frac{d^2z}{dt^2} = Z + R_3.$$

Since the pressure must be perpendicular to the tangent to the instantaneous position of the curve, we do not necessarily require all these equations, though it may be convenient to use them.

534. *Ex.* A helix is constrained to turn about its axis Oz, which is vertical, with a uniform angular velocity ω. Find the motion of a particle of unit mass descending on it under the action of gravity.

Let the axes OA, OB move with the curve and let OA make an angle ωt with some axis of x fixed in space. Let the angle $AON = \theta$. See the figure of Art. 527.

The equations of the helix referred to axes fixed in space are

$$x = a\cos(\theta + \omega t), \qquad y = a\sin(\theta + \omega t), \qquad z = a\theta\tan\alpha;$$

$$\therefore\ 2T = x'^2 + y'^2 + z'^2 = a^2\{(\theta' + \omega)^2 + \tan^2\alpha\theta'^2\}.$$

Substituting in Lagrange's equation, we find after a little reduction

$$a\theta'' = -g\sin\alpha\cos\alpha,$$

which admits of easy integration. It should be noticed that this result is independent of the angular velocity of the guiding curve, provided only it is constant. A similar result holds for any curve on a right circular cylinder turning uniformly about its axis.

To find the pressure of the helix on the particle we use cylindrical coordinates, Art. 491. Let P, Q, R be the components of the pressure, then since in the helix $\rho = a$, $\phi = \theta + \omega t$, we find by substitution

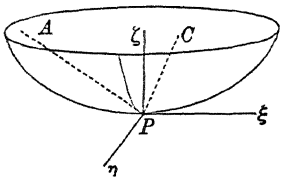
$$P = -a(\theta' + \omega)^2, \qquad Q = a\theta'', \qquad Z - g = a\tan\alpha\theta''.$$

These show that the pressure on the particle is equivalent to a sustaining force $g\cos\alpha$ acting perpendicularly to the osculating plane together with the radial pressure P.

Motion on a Surface.

535. Any Surface. *To find the motion of a particle on a fixed surface.*

Let P be the position of the particle at the time t, $P\eta$ a tangent to the path, $P\zeta$ a normal to the surface, and $P\xi$ that tangent to the surface which is perpendicular to the path. Let PC be the radius of curvature of the path, PA the binormal, then PA, PC lie in the plane $\xi\zeta$. Let χ be the angle $CP\zeta$.



Let X, Y, Z be the resolved impressed forces parallel to any axes x, y, z fixed in space. Let the equation of the surface be $f(x, y, z) = 0$.

The resolved accelerations of the particle in the directions PA, $P\eta$, PC are known to be zero, vdv/ds and v^2/ρ respectively, Art. 496. Hence resolving in the direction $P\eta$,

$$mv\frac{dv}{ds} = X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds},$$

which if U be the work function at once reduces to

$$\tfrac{1}{2}mv^2 = U + C \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

This is the equation of vis viva.

Let R be the pressure of the constraining surface on the particle measured positively inwards. Then resolving along the normal,

$$\frac{mv^2}{\rho}\cos\chi = H + R,$$

where H is the component of the impressed forces. If ρ' be the radius of curvature of the normal section $\eta P\zeta$ of the surface made by a plane through the tangent to the path, it is proved in solid geometry that $\rho = \rho' \cos\chi$. We therefore have

$$\frac{mv^2}{\rho'} = H + R \text{..........................(2)}.$$

536. If a, b are the radii of curvature of the principal sections of the surface at P, ϕ the angle the tangent to the path makes with the section a, we have by Euler's theorem

$$\frac{1}{\rho'} = \frac{\cos^2\phi}{a} + \frac{\sin^2\phi}{b}.$$

Let v_1, v_2 be the resolved velocities of the particle along the tangents to the principal sections, then $v_1 = v\cos\phi$ and $v_2 = v\sin\phi$. The equation (2) then takes the form

$$m\left(\frac{v_1^2}{a} + \frac{v_2^2}{b}\right) = H + R.$$

537. If the forces are conservative, *the velocity of the particle is given by the equation* (1) *in terms of its coordinates at any instant and of the initial conditions.* To determine the velocity at any point we do not require to know the path by which the particle arrived at that point (Art. 181).

The pressure R is given by (2) in terms of the velocity at that point, the normal component of force and the radius of curvature of the normal section of the surface through the tangent. *The pressure is therefore also independent of the path. The whole energy C being given, the pressure depends on the position of the particle and the direction of motion.*

The equation (2) shows that the acceleration of the particle normal to the surface is v^2/ρ'. It is therefore independent of the position of the osculating plane but depends on the direction of motion.

538. To find the path of the particle we resolve in some third direction. Choosing the direction $P\xi$, we have

$$\frac{mv^2}{\rho}\sin\chi = F \text{..........................(3)},$$

where F is the component of the impressed force along that tangent to the surface which is perpendicular to the path. This may also be written in the forms

$$\frac{mv^2}{\rho'}\tan\chi = F, \quad \frac{mv^2}{\rho''} = F,$$

where ρ'' is the radius of curvature of the projection of the path on the tangent plane. It is also called the geodesic radius of curvature.

539. Geodesic path. If the only impressed forces acting on the particle are normal to the surface, $F = 0$, and the third equation shows that either $\sin\chi = 0$ or that the path is a straight line. The path is therefore necessarily a geodesic line.

If the surface is rough, the friction acts opposite to the direction of motion, and F would still be zero. So also if the particle moves in a resisting medium the resistance is opposite to the direction of motion. Generally we conclude that *the path of a particle on a rough surface in a resisting medium when acted on by forces normal to the surface is a geodesic.*

Conversely, *if the path is a geodesic line* we must have $\sin\chi = 0$ and therefore $F = 0$. *The component of the impressed force tangential to the surface must then also be tangential to the path.*

540. *To find the radius of curvature of the path and the position of the osculating plane when the position and direction of motion of the particle are given.*

To effect this we use the two equations

$$\frac{mv^2}{\rho}\sin\chi = F, \quad \frac{1}{\rho'} = \frac{\cos^2\phi}{a} + \frac{\sin^2\phi}{b} = \frac{\cos\chi}{\rho}.$$

The particle being in a given position, v^2, a and b are known. Since ϕ is the angle the direction of motion makes with the principal section whose radius of curvature is a, we have

$$F = A\cos\phi + B\sin\phi,$$

where A and B are the given components of impressed force along the tangents to the principal sections. Thus the values of both $\sin\chi/\rho$ and $\cos\chi/\rho$ follow at once.

541. Motion on a surface of revolution. *When the surface on which the particle moves is one of revolution, it is generally more convenient to use cylindrical coordinates.*

Let the axis of figure be the axis of z and let ξ be the distance of the particle P from that axis. Let the equation of the surface be $z=f(\xi)$. Let U be the work function, and let the mass be unity. The equation of motion obtained by resolving perpendicularly to the plane zOP is

$$\frac{1}{\xi}\frac{d}{dt}(\xi^2\phi')=\frac{dU}{\xi\, d\phi} \quad\text{.......................(1).}$$

We have also the equation of vis viva

$$T=\tfrac{1}{2}\{\xi'^2+z'^2+\xi^2\phi'^2\}=U+C \quad\text{..............(2),}$$

which, by using the equation of the surface, may be written in the form

$$\tfrac{1}{2}\xi'^2\left\{1+\left(\frac{dz}{d\xi}\right)^2\right\}+\tfrac{1}{2}\xi^2\phi'^2=U+C \quad\text{...........(3).}$$

Here accents denote differentiations with regard to the time.

By solving (1) and (3) we determine the two coordinates ξ, ϕ in terms of the time.

In certain cases the solution can be effected. The equation (1) gives

$$\xi^2\phi'\frac{d}{dt}(\xi^2\phi')=\xi^2\frac{dU}{d\phi}\phi'.$$

Let the impressed forces be such that

$$\xi^2 U=F_1(\phi)+F_2(\xi, z) \quad\text{....................(4),}$$

where F_1, F_2 are arbitrary functions. We then have

$$\xi^2\phi'\frac{d}{dt}(\xi^2\phi')=\frac{dF_1(\phi)}{d\phi}\phi'; \qquad \therefore\ \tfrac{1}{2}\xi^4\phi'^2=F_1(\phi)+A \ \text{...(5).}$$

Substituting this value of ϕ' in (3) we find

$$\tfrac{1}{2}\xi^2\xi'^2\left\{1+\left(\frac{dz}{d\xi}\right)^2\right\}=F_2(\xi, z)+C\xi^2-A \quad\text{...........(6).}$$

Since z is a known function of ξ, the variables in this equation are separable. The determination of ξ as a function of t has therefore been reduced to integration. The differential equation of the path is found by dividing (5) by (6). It is evident that here also the variables are separable.

Since the expression for the vis viva, given in (3), can be written in the form

$$T = \tfrac{1}{2}\xi^2 \{P\xi'^2 + \phi'^2\},$$

where P is a function of ξ only, this solution is an example of Liouville's method of solving Lagrange's equations; Art. 522.

542. Motion on a sphere. When the surface on which the particle moves is a sphere, we may use polar coordinates, the centre being the origin. The equations corresponding to (1) and (3) of Art. 541 are found by putting $\xi = l\sin\theta$, where l is the radius; we then have

$$l^2 \frac{d}{dt}(\sin^2\theta\phi') = \frac{dU}{d\phi}, \quad \tfrac{1}{2}l^2\{\theta'^2 + \sin^2\theta\phi'^2\} = U + C.$$

These admit of integration when U, expressed in polar coordinates, has the form

$$\sin^2\theta\, U = F_1(r, \phi) + F_2(r, \theta).$$

The resulting integrals are

$$\left.\begin{aligned} \tfrac{1}{2}l^2\sin^4\theta\,\phi'^2 &= F_1(l, \phi) + A \\ \tfrac{1}{2}l^2\sin^2\theta\,\theta'^2 &= F_2(l, \theta) + C\sin^2\theta - A \end{aligned}\right\}.$$

543. Examples. *Ex.* 1. A particle of mass m moves on the inner surface of a cone of revolution, whose semi-vertical angle is α, under the action of a repulsive force $m\mu/r^3$ from the axis; the angular momentum of the particle about the axis being $m\sqrt{\mu}\tan\alpha$; prove that its path is an arc of a hyperbola whose eccentricity is $\sec\alpha$. [Math. Tripos, 1897.]

Resolve along the generator and take moments about the axis, thus avoiding the reaction, Art. 541. These prove by integration that the path lies on a plane parallel to the axis. The angle between the asymptotes is therefore equal to the angle of the cone.

Ex. 2. A particle P moves on a sphere of radius l under the action both of gravity and a force $X = \mu/x^3$ tending directly from a vertical diametral plane taken as the plane of yz. Show that the determination of the motion can be reduced to integration. If the particle is projected horizontally from the extremity of the axis of x, show that when next moving horizontally, it is in a lower position.

Ex. 3. A particle is acted on by a force the direction of which meets an infinite straight line AB at right angles and the intensity of which is inversely proportional to the cube of the distance from AB. The particle is projected with the velocity from infinity from a point P at a distance a from the nearest point O of the line in a direction perpendicular to OP and inclined at an angle α to the plane AOP. Prove that the particle is always on the sphere the centre of which is O, that it meets every meridian line through AB at the angle α, and that it reaches the line AB in the time $a^2\sec\alpha/\sqrt{\mu}$, where μ is the absolute force.

[Math. Tripos, 1860.]

Ex. 4. A particle moves on a spherical surface of unit radius, its position being determined by its polar distance θ and its longitude ϕ. If the tangential acceleration is always in the meridian, and $\sin^2\theta\, d\phi/dt = h$, $\cot\theta = u$, prove that its value is $h^2(1+u^2)\left(u+\dfrac{d^2u}{d\phi^2}\right)$.

Prove also that the law of force perpendicular to the equatorial plane under which the sphero-conic $\dfrac{1}{\sin^2\theta} = \dfrac{\cos^2\phi}{\sin^2 a} + \dfrac{\sin^2\phi}{\sin^2 b}$ can be described is that of the inverse cube of the distance. [Math. Tripos, 1893.]

Ex. 5. A particle moves on a smooth helicoid, $z = a\phi$, under the action of a force μr per unit mass directed at each point along the generator inwards, r being the distance from the axis of z. The particle is projected along the surface perpendicularly to the generator at a point where the tangent plane makes an angle α with the plane of xy, its velocity of projection being $a\sqrt{\mu}$. Prove that the equation of the projection of its path on the plane of xy is

$$1 + a^2/r^2 = \sec^2\alpha\,\{\cosh(\phi/\cos\alpha)\}^2. \qquad \text{[Math. Tripos, 1896.]}$$

544. Cylinders. *Ex.* 1. A particle moves on a rough circular cylinder under the action of no external forces. Prove that the space described in time t is $\dfrac{a\sec^2\alpha}{\mu}\log\left(1+\dfrac{\mu V\cos^2\alpha t}{a}\right)$ where the particle has initially a velocity V in a direction making an angle α with the transverse plane of the cylinder.

[Math. Tripos, 1888.]

Ex. 2. A heavy particle moves on a rough vertical circular cylinder and is projected horizontally with a velocity V. Prove that at the point where the path cuts the generator at an angle ϕ, the velocity v is given by

$$ag/v^2\sin^2\phi = ag/V^2 + 2\mu\log(\cot\phi + \operatorname{cosec}\phi),$$

and that the azimuthal angle θ and vertical descent z are $ag\theta = \int v^2 d\phi$ and $gz = \int v^2\cot\phi\, d\phi$, the limits being $\phi = \frac{1}{2}\pi$ to ϕ. [Math. Tripos, 1888.]

The cylindrical equations of motion give

$$\frac{d}{dt}(v\sin\phi) = -\frac{\mu}{a}v^2\sin^3\phi, \qquad \frac{d}{dt}(v\cos\phi) = g - \frac{\mu}{a}v^2\sin^2\phi\cos\phi.$$

First eliminating dt and putting $v = 1/w$ we obtain the first result. Secondly eliminating μ we obtain the others.

Ex. 3. A smooth cylinder whose cross section is a cardioid is placed with its generators inclined at an angle α to the vertical and having the generator through the cusp in its highest position, and a particle is projected from the cusp line with velocity V along the inner surface of the cylinder inclined at an angle β to the generator; show that it will leave the surface if $V^2 < \dfrac{6}{5}\dfrac{ag\sin\alpha}{\sin^2\beta}$, where $2a$ is the breadth of any section through the cusp. [Math. Tripos, 1887.]

545. String on a surface. *Ex.* 1. A string, one end of which is fastened at a point of the surface of a smooth circular cylinder whose axis is vertical, winds round the cylinder for part of its length, and terminates in a straight portion of length c at the end of which a particle is tied. Show that when the particle is projected in the direction horizontal and perpendicular to the string it begins to rise or fall according as the velocity is greater or less than $\sin\alpha\,(gc\sec\alpha)^{\frac{1}{2}}$; α being the angle at which the string cuts the generators.

Prove also that during the ensuing motion $\frac{1}{r}\frac{d}{dt}(r^2\omega)+a\omega^2=0$; r being at any time the length of the projection of the straight portion of the string on a horizontal plane, ω the angular velocity of the vertical plane drawn through the string and a the radius of the cylinder. [Coll. Ex. 1895.]

Ex. 2. A string is wound round a vertical cylinder of radius a in the form of a given helix, the inclination to the horizon being i. The upper end is attached to a fixed point on the cylinder, and the lower, a portion of the string of length $l\sec i$ having been unwound, has a material particle attached to it which is also in contact with a rough horizontal plane, the coefficient of friction being μ. Supposing a horizontal velocity V perpendicular to the free portion of the string to be applied to the particle so as to tend to wind the string on the cylinder, determine the motion and prove that the particle will leave the plane after the projection of the unwound portion of the string upon the plane has described an angle

$$\frac{1}{2\mu\tan i}\log\frac{ga}{2\mu V^2\tan^2 i-2\mu gl\tan i+ga}. \qquad \text{[Math. T. 1860.]}$$

Ex. 3. A fine string of length l is fastened to a point A of a smooth cylinder of radius a, and, being wound round the cylinder, has a particle of given mass attached to the free end. Show that, if the particle is projected in any direction, it will, so long as the string is tight and some portion of it remains wound on the cylinder, describe a geodesic line on the surface

$$x\cos\frac{1}{a}\left(\sqrt{l^2-z^2}-\sqrt{x^2+y^2-a^2}\right)+y\sin\frac{1}{a}\left(\sqrt{l^2-z^2}-\sqrt{x^2+y^2-a^2}\right)=a,$$

where the axis of the cylinder is the axis of z, and the axis of x is the radius through A.

Show also that the particle cannot be so projected that the string shall not slip on the cylinder, except when the path lies in the plane of the circular section of the cylinder drawn through A. [Math. Tripos, 1893.]

546. Gauss' coordinates. The motion of a particle on a surface may also be investigated by using the geodesic polar coordinates of Gauss. In this method every surface has a geometry of its own, in which all the lines under consideration are drawn on the surface. The geodesics on the surface correspond to straight lines on a plane, and the properties of the figures are discussed by reasoning analogous to that of two dimensions.

Let O be any origin, ρ the length of the geodesic drawn from O to any moving point P. Let ω be the angle OP makes with some fixed geodesic Ox. Let OP' be a neighbouring geodesic, PL the perpendicular to OP'. Then in the limit $LP'=d\rho$, $PL=Pd\omega$. The theorem that $OP=OL$ is proved in Salmon's *Solid Geometry*, Art. 394, edition of 1882. The quantity P is a function of ρ and ω, whose form depends on the particular surface under consideration. On a plane $P=\rho$, and on a sphere of radius a, $P=a\sin\rho/a$. On an ellipsoid when the origin O is at an umbilicus, $P=y\operatorname{cosec}\omega$, where ω is the angle the geodesic OP makes with the arc containing the four umbilici. The difficulty of finding the value of P for any surface prevents this method from coming into general use.

The vis viva $2T$ of a particle of unit mass is given by

$$T=\tfrac{1}{2}(\rho'^2+P^2\omega'^2),$$

where accents as usual denote differential coefficients with regard to the time.

Let U be the work function; F, G the accelerations at P along and perpendicular to the geodesic radius vector OP. We have by Lagrange's theorems,

$$\frac{d}{dt}\frac{dT}{d\rho'} - \frac{dT}{d\rho} = \frac{dU}{d\rho} = F; \qquad \therefore\ F = \rho'' - P\frac{dP}{d\rho}\omega'^2. \ldots\ldots\ldots\ldots (1).$$

$$\frac{d}{dt}\frac{dT}{d\omega'} - \frac{dT}{d\omega} = \frac{dU}{d\omega} = PG; \qquad \therefore\ G = \frac{1}{P}\frac{d}{dt}(P^2\omega') - \frac{dP}{d\omega}\omega'^2.$$

Since $\dfrac{dP}{dt} = \dfrac{dP}{d\rho}\rho' + \dfrac{dP}{d\omega}\omega'$, this reduces to

$$G = P\omega'' + \frac{dP}{d\omega}\omega'^2 + 2\frac{dP}{d\rho}\omega'\rho' \ldots\ldots\ldots\ldots (2).$$

547. *We may also arrive at these results without using Lagrange's equations.* Let u, v be the component velocities of P along and perpendicular to the tangent PT at P to the geodesic OP. Let $P'T'$ be the projection of the tangent to OP' on the tangent plane at P. Since the tangent planes at P, P' make an indefinitely small angle with each other the component velocities at P along and perpendicular to $P'T'$ are $u+du$ and $v+dv$. If $d\theta$ be the angle PT makes with $P'T'$, the accelerations along and perpendicular to PT are (as in Art. 225),

$$F = \frac{du}{dt} - v\frac{d\theta}{dt}, \qquad G = \frac{dv}{dt} + u\frac{d\theta}{dt}.$$

Now $u=\rho'$, $v=P\omega'$, and by a theorem proved in Salmon's *Solid Geometry*, Art. 392, $d\theta = \dfrac{dP}{d\rho}d\omega$. We therefore have

$$F = \rho'' - P\omega'^2\frac{dP}{d\rho}, \qquad G = \frac{d}{dt}(P\omega') + \rho'\omega'\frac{dP}{d\rho}.$$

These reduce to the same forms as before.

548. *Ex.* A particle P, constrained to move on an ellipsoid, is attached to an umbilicus by a string of given length, which also lies on the surface. Prove that the particle describes a geodesic circle with a uniform velocity V, and that the angular velocity of the string about the umbilicus is $V\sin\omega/y$. Prove also that the accelerating tension is $V^2\cos\beta/y$, where β is the angle the tangent at P to the string makes with the axis of y.

549. Developable surfaces. When the surface on which the particle moves is developable, we may sometimes fix the position of the particle by using the edge as a curve of reference. Let s be the arc of the edge measured from some fixed point A to a point Q such that the tangent at Q passes through P. Let $QP=u$ measured positively in the same direction as s. We then have

$$v^2 = \frac{u^2}{\rho^2}s'^2 + (u'+s')^2.$$

The form of the surface being given, the radius of curvature ρ of the edge at Q is known as a function of s. When U is given as a function of u and s the Lagrangian method supplies two equations to find the coordinates u and s.

Ex. A heavy particle moves on a developable surface whose edge is a helix with its axis vertical. Obtain two integrals by which s' and u' may always be found in terms of u and s. Show also that if the particle is projected along a tangent to the helix, it will continue to describe that tangent.

Motion of a heavy particle on a surface of revolution.

550. *To find the motion of a heavy particle on a surface of revolution the axis of which is vertical.*

Let the axis of z be the axis of the surface and let z be measured upwards. The velocity v is then given by

$$v^2 = 2g(h-z) \quad\quad (1),$$

where h is a constant depending on the initial conditions. Let the plane $z=h$ be called *the level of no velocity.*

Let ξ be the distance of the particle P from the axis of figure, and ϕ the angle the plane zOP makes with the plane zOx. Then

$$\xi^2 \frac{d\phi}{dt} = A \quad\quad (2),$$

where mA is the constant angular momentum and its value is known when the initial values of ξ and $d\phi/dt$ are given; Art. 492. The velocity v at any point being given by (1), the angular momentum A must lie between zero and $v\xi$. It is the former when the particle is moving in the plane zOP and the latter when moving horizontally. The particle therefore can occupy only those points of the surface at which $v\xi > A$, i.e. those points at which $2g(h-z)\xi^2 > A^2$. If then we describe the cubic surface

$$(h-z)\xi^2 = A^2/2g \quad\quad (3),$$

the ξ of the particle for any value of z must be greater than the corresponding ξ of the cubic surface.

This cubic divides the given surface of revolution into zones, separated by horizontal circles, and the particle can move only in those zones which are more remote from the axis of figure than the corresponding portions of the cubic. The zone actually moved in is determined by the point of projection. The particle moves round the axis of figure and must continue to ascend or to descend until it arrives at a point at which the vertical velocity can be zero, that is, until it reaches one of the boundaries of the zone.

If the particle is projected horizontally it is on the boundary of two zones. It will move on that neighbouring zone which is the more remote from the axis than the corresponding portion of the cubic. If the cubic touch the surface of revolution, the particle is situated on an evanescent zone and will then describe

a horizontal circle. The path is stable or unstable according as the neighbouring zones are less or more remote from the axis of figure than the cubic surface.

551. *Ex. A particle is projected horizontally with a velocity V at a point whose coordinates are ξ, z. Will it rise or fall?*

If mR be the pressure on the particle, ψ the angle the radius of curvature makes with the vertical, we see by resolving vertically, that the particle if inside and $\psi < \frac{1}{2}\pi$ will rise or fall according as $R \cos\psi$ is greater or less than g.

To find R we resolve along the normal to the surface. Since the particle is moving along that principal section whose radius of curvature is the normal n, we have $V^2/n = R - g\cos\psi$, Art. 536. Since $n \sin\psi = \xi$, we see that *the particle will rise, fall, or describe a horizontal circle according as V^2 is greater, less, or equal to $g\xi \tan\psi$.* If $z = f(\xi)$ be the equation of the surface of revolution, $\tan\psi = dz/d\xi$.

To find the level to which the particle will rise or fall we use the cubic surface described in Art. 550, the constants A and h being known from the equations $V\xi = A$, $V^2 = 2g(h-z)$. The intermediate motion may be deduced from the equations (1), (2) of the same article.

552. *Ex. To find the pressure on the particle when in any position.*

We use the formula given in Art. 536. The principal radii of curvature of the surface are the radius of curvature ρ of the meridian and the normal n. The velocity perpendicular to the meridian being $v_2 = \xi d\phi/dt$, the velocity v_1 along the meridian is given by $v^2 = v_1^2 + v_2^2$. The formula

$$\frac{v_1^2}{\rho} + \frac{v_2^2}{n} = R - g\cos\psi,$$

shows that

$$R = g\cos\psi + \frac{2g(h-z)}{\rho} + \frac{A^2}{\xi^2}\left(\frac{1}{n} - \frac{1}{\rho}\right).$$

This problem has a special interest because we can use it to represent experimentally the path of a particle under the action of a centre of force. If Q be the projection of the particle on a horizontal plane, the motion of Q is the same as that of a particle moving under the action of a central force whose magnitude is $R \sin\psi$. If then a surface is so constructed that the generating curve satisfies the differential equation $R\sin\psi = \mu/\xi^2$, where R has the value given above, the path of Q should be a conic with a focus at the origin.

The experiment cannot be properly tried with a particle, for the surface must then be very smooth. It is better to replace the particle by a small sphere which is made to roll on a rough surface, but in that case, the theory must be modified to allow for the size of the particle. *Nature*, 1897.

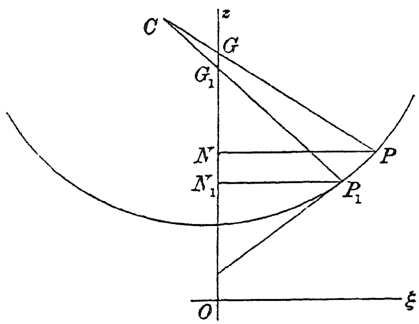
553. Small oscillation. *Ex. A heavy particle P, describing a horizontal circle on a surface of revolution, is slightly disturbed. It is required to find the oscillations to a first approximation.*

The plane zOP may be reduced to rest if we apply to the particle a horizontal acceleration $\xi(d\phi/dt)^2$, Art. 495. Since $\xi^2 d\phi/dt = A$, this acceleration is equal to A^2/ξ^3. Resolving along the meridian, we have

$$\frac{d^2s}{dt^2} = \frac{A^2}{\xi^3}\cos\psi - g\sin\psi,$$

where ψ is the angle PGO which the normal to the surface makes with the axis.

Let the radius of the mean circle be $N_1P_1=c$ and let the normal to the surface at any point of its circumference make an angle $P_1G_1O=\gamma$ with the vertical.



Since s may be taken to be the arc of the meridian between the particle and the mean circle, we have

$$\xi=c+s\cos\gamma, \qquad \psi=\gamma+s/\rho,$$

where ρ is the radius of curvature of the meridian at its intersection with the mean circle.

Substituting, we find by Taylor's theorem $\dfrac{d^2s}{dt^2}=F-p^2s$,

$$F=\frac{A^2}{c^3}\cos\gamma-g\sin\gamma, \qquad p^2=\frac{A^2\sin\gamma}{c^3\rho}+\frac{3A^2\cos^2\gamma}{c^4}+\frac{g\cos\gamma}{\rho}.$$

The position of the circle of reference is as yet arbitrary except that the deviation s must be small. Let it be so chosen that the mean value of s (taken for any long time) is zero; we then have $F=0$. The mean circle and the angular momentum mA are so related that $A^2=c^3g\tan\gamma$, while the oscillatory motion is given by $s=L\sin(pt+M)$ where L, M are the constants of integration.

To find the motion round the axis of figure we use the equation $\xi^2d\phi/dt=A$;

$$\therefore\ \frac{d\phi}{dt}=\frac{A}{\xi^2}=\frac{A}{c^2}\left(1-\frac{2s}{c}\cos\gamma\right);$$

$$\therefore\ \phi=\frac{At}{c^2}+\frac{2A\cos\gamma}{c^3}\cdot\frac{L}{p}\cos(pt+M)+N,$$

where N is the constant of integration.

If we write ω for the mean value of $d\phi/dt$, we have $A=c^2\omega$. We then find

$$c\omega^2=g\tan\gamma, \qquad p^2=\omega^2\left(\frac{c\sin\gamma}{\rho}+3\cos^2\gamma\right)+\frac{g\cos\gamma}{\rho}.$$

The time the particle takes to travel from the highest position to the lowest or the reverse is π/p.

554. The Paraboloid. *Ex.* 1. A smooth paraboloid is placed with its axis vertical and vertex downwards, and its equation is $\xi^2=4az$. A heavy particle is projected horizontally with velocity V, the initial altitude being $z=b$, show that the particle is again moving horizontally at an altitude $z=V^2/2g$. Show also that the pressure on the surface at any point of the path is inversely proportional to the radius of curvature of the parabola.

To prove the first, we notice that the angular momentum $A=V\xi$ where $\xi^2=4ab$. The cubic $\xi^2(h-z)=A^2/2g$ becomes $z^2-hz+V^2b/2g=0$, one root of the quadratic being $z=b$, the other b' is given either by $b+b'=h$ or $b'=V^2/2g$. The second part follows from Art. 552.

If the time T of passing from one limit to the other be required, we first notice that

$$v^2=\left\{\left(\frac{d\xi}{dz}\right)^2+1\right\}\left(\frac{dz}{dt}\right)^2+\frac{A^2}{\xi^2}=2g(h-z);$$

$$\therefore \sqrt{2g}\,T=\int\frac{\sqrt{(a+z)}\,dz}{\sqrt{(z-b)(b'-z)}},$$

the limits being b and b'. This integral can be reduced to elliptic forms by putting

$$a+z=(b'+a)\cos^2\theta.$$

Ex. 2. A particle moves under the action of gravity on a smooth paraboloid whose axis is vertical, vertex downwards and latus rectum $4a$. If the particle be projected along the surface in the horizontal plane through the focus with a velocity $\sqrt{(2nag)}$, prove that the initial radius of curvature ρ of the path, and the angle θ which the radius of curvature makes with the axis, are given by

$$\sqrt{(n^2+1)}\,\rho=2na\sqrt{2},\qquad (1-n)\tan\theta=1+n. \qquad \text{[Math. T. 1871.]}$$

Ex. 3. A heavy particle moves on a paraboloid with its axis vertical, the equation of the surface being $x^2/\alpha+y^2/\beta=4z$. Show that the particle when moving horizontally must lie on the quartic surface $\frac{\alpha\beta}{4}\left(\frac{1}{p^2}-4\right)\left(\frac{B}{2g}-\frac{1}{p^2}\right)=\frac{z(h-z)}{p^2}$, where $\frac{1}{p^2}=\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+4$, and B is the initial value of $\frac{1}{p^2}\left(\frac{x'^2}{\alpha}+\frac{y'^2}{\beta}+2g\right)$. Show also that when the paraboloid is a surface of revolution, the intersection reduces to two horizontal planes and two coincident planes at the vertex.

555. The Conical Pendulum. *To find the motion of a heavy particle P on a smooth sphere**.

It will be convenient in this problem to take the origin of coordinates at the centre O of the sphere and to measure Oz vertically *downwards*. Let l be the length of the string OP and θ the angle it makes with Oz. Let ϕ be the angle the vertical plane zOP makes with some fixed plane zOx. Let r be the

* The problem of the conical pendulum has been considered by Lagrange in the second volume of his *Mécanique Analytique*. He deduces equations equivalent to (1) and (3) of Art. 555 from his generalized equations, and notices that the cubic has three real roots. He reduces the determination of t and ϕ to integrals, and makes approximations when the bounding planes are close together. He refers also to a memoir of Clairaut in 1735. There is an elaborate memoir by Tissot in *Liouville's Journal*, vol. XVII. 1852. He expresses t, z, ϕ and the arc s in elliptic integrals in terms of u. A long communication by Chailan may be found in the *Bulletin de Soc. Math. de France*, 1889, vol. XVII. There is a brief discussion of this problem in Greenhill's *Applications of Elliptic Functions*, 1892, Art. 208.

distance of P from Oz. Let h be the altitude above O of the level of zero velocity. We now proceed as in Art. 550.

By the principles of angular momentum and vis viva,

$$r^2 \frac{d\phi}{dt} = A, \quad v^2 = l^2 \left(\frac{d\theta}{dt}\right)^2 + r^2 \left(\frac{d\phi}{dt}\right)^2 = 2g(h + l\cos\theta) \ldots (1).$$

Eliminating $d\phi/dt$ and writing $r = l\sin\theta$,

$$l^2 \sin^2\theta \left(\frac{d\theta}{dt}\right)^2 = 2g(h + l\cos\theta)\sin^2\theta - \frac{A^2}{l^2} \ldots\ldots\ldots (2).$$

Putting $z = l\cos\theta$, this may also be written in the form

$$l^2 \left(\frac{dz}{dt}\right)^2 = 2g(h + z)(l^2 - z^2) - A^2 \ldots\ldots\ldots\ldots\ldots (3).$$

To find the positions of the horizontal sections between which the particle oscillates (Art. 550), we put $dz/dt = 0$. We thus have the cubic

$$(h + z)(l^2 - z^2) - A^2/2g = 0 \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Since the initial value of z must make $(dz/dt)^2$ positive, the left-hand side of the cubic (4) is positive for some value of z lying between $z = \pm l$. When $z = \pm l$ the left-hand side is negative, hence the cubic has two real roots lying between $\pm l$ and separated by the initial value of z. Let these roots be $z = a$ and $z = b$. Lastly when z is very large and negative the left-hand side is positive, the third root of the cubic is therefore negative and numerically greater than l. Let this root be $z = -c$. *The particle oscillates between the two horizontal planes defined by $z = a$, $z = b$.*

Since the cubic can be written in the form

$$z^3 + hz^2 - l^2 z + (A^2/2g - l^2 h) = 0,$$

we have the obvious relations

$$a + b - c = -h, \quad (a + b)c - ab = l^2, \quad abc = A^2/2g - l^2 h.$$

Conversely, when the depths a and b of the two boundaries of the motion are given, the values of the other constants of the motion, viz. c, h, and A, follow at once. We have

$$c = \frac{l^2 + ab}{a + b}, \quad \frac{A^2}{2g} = \frac{(l^2 - a^2)(l^2 - b^2)}{a + b}, \quad h = c - a - b.$$

556. *Ex.* Prove (1) that one of the two horizontal planes bounding the motion lies below the centre; (2) that the plane equidistant from the two bounding planes also lies below the centre; (3) that both the bounding planes lie below the centre if $2ghl^2 < A^2$; (4) if a length $OC = c$ be measured upwards from the centre O,

the point C is not only above the top of the sphere, but above the level of zero velocity.

To prove (1), we notice that if all the roots were negative, every coefficient of the cubic (4) would be positive, which is not the case. To prove (2); since both a and b are numerically less than l, it follows from the value of c that $a+b$ is positive. (3) The two roots a and b will have the same or different signs according as the left-hand side of the cubic when $z=0$ and $z=l$ has the same or different signs. The fourth result follows from the fact that $c-h$, i.e. $a+b$, is positive.

The first and third results follow also from Descartes' rule of signs; for since all the roots of the cubic are real, there are as many positive roots as changes of sign, and as many negative roots as continuations.

557. *Ex.* To find *the tension of the string* we produce the radius vector OP outwards to a point Q so that PQ is half the length of the string. Let z' be the depth of Q below the level of zero velocity. Prove that the tension mR is given by $lR=2gz'$. Thence show that *the string can become slack only when Q crosses the level of zero velocity.* It may be noticed that the tension or pressure on a sphere is independent of the angular momentum mA.

558. *Ex.* 1. *A particle P is projected horizontally with a velocity V. Determine whether it will rise or fall, and find the position of the other boundary to the motion.*

Let the initial radius OP make an angle α with the vertical. Resolving along the normal, we find that the initial tension mR is given by $R=g\cos\alpha+V^2/l$. The particle will rise or fall according as $R\cos\alpha$ is $>$ or $<g$, that is, according as $V^2\cos\alpha$ is $>$ or $<lg\sin^2\alpha$. If these are equal the particle describes a horizontal circle. See Art. 551.

To determine how far it will rise or fall, we notice that one root of the cubic in Art. 555 is known, viz. $z=l\cos\alpha$; the cubic may therefore be reduced to a quadratic. But *it is more easy to repeat the reasoning.* We have by the principles of angular momentum and vis viva

$$r^2\frac{d\phi}{dt}=Vl\sin\alpha, \qquad l^2\left(\frac{d\theta}{dt}\right)^2+r^2\left(\frac{d\phi}{dt}\right)^2=V^2+2gl(\cos\theta-\cos\alpha).$$

Eliminating $d\phi/dt$ and putting zero for $d\theta/dt$, the limiting values of θ are found from

$$V^2\frac{\sin^2\alpha}{\sin^2\theta}=V^2+2gl(\cos\theta-\cos\alpha);$$

$$\therefore\ V^2(\cos\theta+\cos\alpha)=2gl\sin^2\theta.$$

Putting $V^2/2gl=2n$ for brevity, we find

$$\cos\theta=-n+\sqrt{(1-2n\cos\alpha+n^2)},$$

where the positive sign is given to the radical because $\cos\theta$ must be less than unity. This value of $\cos\theta$ and $\cos\theta=\cos\alpha$ determine the positions of the bounding planes of the motion.

Ex. 2. A heavy particle, constrained to move on the surface of a smooth sphere of radius a, is projected horizontally with a velocity V from a point on the surface whose depth below the centre is x. Prove that, when next moving horizontally, the depth x' of the particle below the same point is given by

$$2g(x'^2-a^2)+V^2(x'+x)=0.$$

Ex. 3. In the centre of a hollow sphere resides a repulsive force. A heavy particle is projected horizontally along the surface of the sphere from a point distant 60° from the highest point with a velocity due to falling through the diameter by its weight only. Prove that it will be again moving horizontally at a point whose distance from the lowest point is $\tan^{-1}\sqrt[4]{\frac{3}{4}}$. [Coll. Ex.]

Ex. 4. A particle is attached by a string to the top of a hemispherical dome, and is projected horizontally along the interior surface, which is rough, with a velocity just sufficient to prevent it from at once leaving the surface. Find the velocity after describing a given arc, and show that it will always remain in contact with the surface. [Math. Tripos, 1853.]

559. *Ex.* 1. *Show that the radius of curvature of the path and the inclination* χ *of the osculating plane to the normal to the sphere are given by*

$$\frac{l^2}{\rho^2}=1+\frac{g^2A^2}{v^6}, \qquad \tan\chi=\frac{gA}{v^3},$$

where v is the velocity and mA the constant angular momentum.

We follow the method given in Art. 540. Let F be the component of acceleration along that tangent to the sphere which is perpendicular to the direction of motion. Then $\frac{\cos\chi}{\rho}=\frac{1}{l}$, $\frac{v^2}{\rho}\sin\chi=F$. To find F, we notice that the acceleration perpendicular to the meridian plane is zero, while that tangential is $g\sin\theta$. Hence if the direction of motion makes an angle ψ with the meridian,

$$F=g\sin\theta\sin\psi.$$

Since the components of velocity in and perpendicular to the meridian plane are $a\theta'$ and $a\sin\theta\phi'$, we have $v\cos\psi=a\theta'$, $v\sin\psi=l\sin\theta\phi'$. Choosing the latter component to find ψ and remembering that $l^2\sin^2\theta\phi'=A$, the values of $\cos\chi/\rho$ and $\sin\chi/\rho$ are evident.

Ex. 2. A particle is projected with velocity V horizontally from a point on the surface of a smooth sphere. Prove that the radius of curvature of its path is $\frac{lV^2}{\sqrt{(V^4+l^2g^2\sin^2\alpha)}}$ where l is the radius of the sphere and α the inclination to the vertical of the radius at the point. [Coll. Ex. 1881.]

Ex. 3. A particle is projected inside a smooth sphere of radius l with a velocity $\sqrt{2gl}$ along a tangent to the horizontal equator, prove that at first the radius of curvature is $2l/\sqrt{5}$. [Coll. Ex. 1897.]

560. *Ex.* Prove that *the projection of the path of the particle on a horizontal plane is a central orbit described under a force* $R\sin\theta=\frac{gr}{l^2}\{2h+3\sqrt{(l^2-r^2)}\}$, *where the radical changes sign when* $r=l$.

Show also that if the two roots a and b of the cubic in Art. 555 have the same signs, the central path is a spiral curve touching alternately two circles whose radii are $\sqrt{(l^2-a^2)}$ and $\sqrt{(l^2-b^2)}$, the curve being always concave to the centre of force. If a and b have opposite signs the central path after touching each bounding circle, touches the circle $r=l$ and then touches the other bounding circle. There will be a point of inflexion only if R vanishes and changes sign.

561. *Ex.* *If we write* $\frac{1}{3}h + l\cos\theta = \kappa\cos\phi$, *the general equation of motion of a conical pendulum may be reduced to the form*

$$-\sin^2\phi\left(\frac{d\phi}{dt}\right)^2 = \frac{\kappa g}{2l^2}(\cos 3\phi - \cos 3a),$$

by properly choosing the constants κ *and* a.

Show that these values are

$$\kappa = \frac{2}{3}\sqrt{(h^2+3l^2)}, \qquad -\kappa^3\cos 3a = \frac{2A^2}{g} + \frac{8}{27}h^3 - \frac{8}{3}hl^2.$$

Find also the positions of the bounding planes when the constants κ and a of the motion are given.

562. Time of passage. *The motion of the particle as it travels from one boundary to the other may be found by an elliptic integral.*

We write the equation (3) of Art. 555 in the form

$$\frac{\sqrt{(2g)}}{l}t = \int\frac{dz}{\sqrt{(a-z)(z-b)(z+c)}},$$

where the limits are $z=a$ and $z=b$, and $a>b$. Putting $z=a-\xi^2$, the integral takes a standard form which is reduced to an elliptic integral by writing $\xi = \sin\psi\sqrt{(a-b)}$, i.e. we write

$$z = a\cos^2\psi + b\sin^2\psi;$$

$$\therefore\ \frac{\sqrt{(2g)}}{l}t = \frac{2}{\sqrt{(a+c)}}\int\frac{d\psi}{\sqrt{(1-\kappa^2\sin^2\psi)}},$$

where
$$\kappa^2 = \frac{a-b}{a+c}, \qquad c = \frac{l^2+ab}{a+b}.$$

If the time of passage from one boundary to the other is required, the limits are 0 and $\frac{1}{2}\pi$.

If the two bounding planes are close together, κ is small. By expanding in powers of κ and effecting the integrations we find that the time from one boundary to the other is given by

$$\frac{\sqrt{(2g)}}{l}t = \frac{\pi}{\sqrt{(a+c)}}\left\{1 + (\tfrac{1}{2})^2\kappa^2 + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2\kappa^4 + \&c.\right\}.$$

If the two bounding planes are also close to the lowest point, we put

$$a = l\cos\alpha = l(1-\tfrac{1}{2}\alpha^2), \quad b = l\cos\beta = l(1-\tfrac{1}{2}\beta^2).$$

We then find that the time of passage from one boundary to the other is

$$t = \frac{\pi}{2}\sqrt{\frac{l}{g}}\left(1 + \frac{\alpha^2+\beta^2}{16}\right),$$

the fourth powers of α and β being neglected. This result is given by Lagrange.

Let $u = \int_0^{\psi} \frac{d\psi}{\sqrt{(1-\kappa^2 \sin^2\psi)}}$ and K be the value of u when $\psi = \frac{1}{2}\pi$. Let t be the time of passage from the lower boundary to the depth z defined by any value of ψ, and T the time from one boundary to the other, then $t/T = u/K$.

563. *Ex.* 1. Prove that when half the time of passing from the lower to the upper boundary has elapsed, the particle is above the mean level between the two boundaries. Prove also that the depth of the particle is then $(\kappa' a + b)/(\kappa' + 1)$, where $\kappa'^2 = 1 - \kappa^2$. [Tissot.]

Ex. 2. Prove that when a quarter of the time has elapsed, the depth z of the particle is

$$z = \frac{a\sqrt{\kappa'}\,(\sqrt{(1+\kappa')}+1) + b\,(\sqrt{(1+\kappa')}-\sqrt{\kappa'})}{(1+\sqrt{\kappa'})\sqrt{(1+\kappa')}}.$$

564. The apsidal angle. *To find the change in the value of ϕ as the particle moves from one bounding plane to the other.*

Eliminating dt between (1) and (3) of Art. 555 we find

$$\frac{\sqrt{(2g)}}{Al}\phi = \int \frac{dz}{\sqrt{(a-z)}\sqrt{(z-b)}\sqrt{(z+c)(l^2-z^2)}},$$

where the limits of integration are $z = b$ and $z = a$, and $a > b$. Putting $a = m + \mu$, $b = m - \mu$, $z = m + \xi$ so that m is the middle value of z and μ the extreme deviation on each side of the middle, we have

$$\frac{\sqrt{(2g)}}{Al}\phi = \int \frac{d\xi}{\sqrt{(\mu^2-\xi^2)}\sqrt{(m+c+\xi)\{l^2-(m+\xi)^2\}}},$$

where the limits are $\xi = -\mu$ and μ.

565. When the *bounding planes are close to each other*, the range μ of the values of ξ is small. If also the planes are not near the lowest point, the two last factors in the denominator are not small for any value of ξ. We may therefore expand these in powers of ξ and thus put the integral into the form

$$\phi = \int \frac{d\xi}{\sqrt{(\mu^2-\xi^2)}}(P + Q\xi + R\xi^2) = \pi(P + \tfrac{1}{2}R\mu^2).$$

After calculating P and R, this gives

$$\phi = \frac{\pi l}{\sqrt{(l^2+3m^2)}}\left\{1 - \frac{3(3l^2+13m^2)m^2\mu^2}{4(l^2-m^2)(l^2+3m^2)^2}\right\}.$$

566. *If both the bounding planes are near the lowest point of the sphere*, l and z are nearly equal, and the last factor in the denominator of ϕ (Art. 564), may be so small that its changes in value are considerable fractions of itself. We write the integral in the form

$$\frac{\sqrt{(2g)}}{Al}\phi=\int\frac{dz}{\sqrt{(a-z)}\sqrt{(z-b)}(l-z)}\cdot\frac{1}{\sqrt{(c+z)(l+z)}}.$$

The two factors in the denominator of the second fraction are not small and these may be expanded in powers of some small quantity properly chosen. We shall make the expansion in powers of $l-z=\eta$.

Remembering the values of A and c found in Art. 555, we have

$$\phi=\frac{1}{2}\sqrt{(l-a)}\sqrt{(l-b)}\int\frac{dz}{\sqrt{(a-z)}\sqrt{(z-b)}(l-z)}\left\{1+\frac{1}{2}\frac{\eta}{c+l}+\frac{\eta}{2l}\right\};$$

all these integrals are common forms. To find the first we put $l-z=1/u$. We have

$$\int\frac{dz}{\sqrt{(a-z)}\sqrt{(z-b)}(l-z)}=\frac{1}{\sqrt{(l-a)}\sqrt{(l-b)}}\int\frac{du}{\sqrt{(\alpha-u)}\sqrt{(u-\beta)}},$$

where α and β are two constants which we need not calculate. For since the limits of the first integral, viz. $z=a$, $z=b$, make the denominator vanish, the limits of the other must be $u=\alpha$, $u=\beta$. Putting $u=\frac{1}{2}(\alpha+\beta)+\xi$ we see at once that the value of that integral is π. Since $\eta=l-z$ the values of the remaining integrals have just been found. Hence

$$\phi=\frac{\pi}{2}\left\{1+\frac{1}{2}\sqrt{(l-a)}\sqrt{(l-b)}\left(\frac{a+b}{(l+a)(l+b)}+\frac{1}{l}\right)\right\},$$

where we have written for $c+l$ its value given in Art. 555.

If p, q be the radii of the circles which bound the oscillation, we have

$$l-a=\frac{p^2}{2l},\qquad l-b=\frac{q^2}{2l},$$

and in the small terms which contain the product pq as a factor, we can write $a=l$, $b=l$; hence (see Art. 562)

$$\phi=\frac{\pi}{2}\left\{1+\frac{3}{8}\frac{pq}{l^2}\right\},\qquad t=\frac{\pi}{2}\sqrt{\frac{l}{g}}\left\{1+\frac{1}{16}\frac{p^2+q^2}{l^2}\right\}.$$

The first of these results differs from that given by Lagrange. The correction was first made by M. Bravais in a note to the *Mécanique Analytique*.

567. *Ex.* A simple spherical pendulum of length l is drawn out to the horizontal position and is then projected horizontally with a velocity $2pl$. Show that, if θ is the angle that the string makes with the vertical, and ϕ the azimuthal angle of the vertical plane through the string, $\sin\theta\sin(\phi-pt)=\frac{p}{n}\sqrt{2\cos\theta}$, where n is equal to $\sqrt{g/l}$. [Math. Tripos, 1893.]

Motion on an Ellipsoid.

568. Cartesian coordinates. *To find the motion of a particle of unit mass on an ellipsoid*.*

Let X, Y, Z be the components of the impressed forces in the directions of the principal axes. Let R be the pressure on the particle measured positively inwards. Since the direction cosines of the normal are px/a^2, &c., the equations of motion are

$$x'' = X - Rp\frac{x}{a^2}, \qquad y'' = Y - Rp\frac{y}{b^2}, \qquad z'' = Z - Rp\frac{z}{c^2} \ldots(1),$$

where accents denote differential coefficients with regard to the time. We also have from the equation of the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad \frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 0 \ldots\ldots\ldots\ldots(2),$$

$$\frac{xx''}{a^2} + \frac{yy''}{b^2} + \frac{zz''}{c^2} + \frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} = 0 \ldots\ldots\ldots\ldots(3).$$

Multiplying the dynamical equations (1) by x', y', z', adding and integrating, we have

$$\tfrac{1}{2}(x'^2 + y'^2 + z'^2) = C + \int (X\,dx + Y\,dy + Z\,dz) \ldots\ldots(4);$$

$$\therefore\ \tfrac{1}{2}v^2 = C + U,$$

where U is the work function and C is a constant. This is of course the equation of vis viva.

Substituting from (1) in (3), we find

$$Rp\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right) = \left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2}\right) + \left(\frac{Xx}{a^2} + \frac{Yy}{b^2} + \frac{Zz}{c^2}\right) \ldots(5).$$

* The motion of a particle constrained to remain on an ellipsoid is discussed by Liouville in his *Journal*, vol. XI. 1846. He uses elliptic coordinates and shows that the variables can be separated when $U(\mu^2 - \nu^2) = F_1(\mu) - F_2(\nu)$. There is also a paper on the same subject by W. R. Westropp Roberts in the *Proceedings of the Mathematical Society*, 1883. He also uses elliptic coordinates and especially treats of the case in which the path is a line of curvature. The case in which the particle is attracted to the centre by a force proportional to the distance is solved in Cartesian coordinates by Painlevé, *Leçons sur l'intégration des équations différentielles de la Mécanique*, 1895. He also treats separately the limiting case of a heavy particle moving on a paraboloid whose axis is vertical. There is a short paper by T. Craig in the *American Journal of Mathematics*, vol. I. 1878. He discusses the same problem as Painlevé, beginning with Cartesian coordinates, but passing quickly to Elliptic coordinates. He shows that the path is a geodesic when the central force is zero and the particle is acted on by what is equivalent to a force tangential to the path and varying as $f(t) + F(s)v$ where s is the arc described. This result follows also from Art. 539.

In an ellipsoid we have

$$\frac{1}{p^2}=\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}, \qquad \frac{1}{D^2}=\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2} \quad\text{........(6)},$$

where D is the semi-diameter of the ellipsoid whose direction cosines are (l, m, n). Also the radius of curvature of the normal section whose tangent is parallel to D is $\rho=D^2/p$. Taking D to be parallel to the tangent to the path $l=x'/v$, $m=y'/v$, $n=z'/v$. The equation (5) is therefore the Cartesian equivalent of

$$R=\frac{v^2}{\rho}-N \quad\text{........................(7)},$$

where N is the *inward* normal component of the impressed force.

569. *In certain cases we may find another integral.* Differentiating (5) and remembering (6), we have

$$\frac{d}{dt}\left(\frac{R}{p}\right)=2\left(\frac{x'x''}{a^2}+\frac{y'y''}{b^2}+\frac{z'z''}{c^2}\right)+\frac{d}{dt}\left(\frac{Xx}{a^2}+\frac{Yy}{b^2}+\frac{Zz}{c^2}\right).$$

Substituting for x'', y'', z'' from (1) and using (6),

$$2Rp\left(-\frac{p'}{p^3}\right)+\frac{d}{dt}\left(\frac{R}{p}\right)=\frac{1}{a^2}\left(2x'X+\frac{dXx}{dt}\right)+\&c.,$$

$$\therefore\ p^2\frac{d}{dt}\left(\frac{R}{p^3}\right)=\frac{1}{a^2x^2}\frac{d}{dt}(Xx^3)+\&c. \quad\text{..............(8)}.$$

If then the forces acting on the particle are such that

$$\frac{1}{a^2x^2}\frac{d}{dt}(Xx^3)+\frac{1}{b^2y^2}\frac{d}{dt}(Yy^3)+\frac{1}{c^2z^2}\frac{d}{dt}(Zz^3)=0 \quad\text{......(9)},$$

we have

$$R=Ap^3 \quad\text{........................(10)}.$$

Substituting in (5) or (7), we have the third integral which may be written in either of the forms

$$\left.\begin{aligned}\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}+\frac{Xx}{a^2}+\frac{Yy}{b^2}+\frac{Zz}{c^2}&=Ap^2\\ \frac{v^2}{\rho}-N&=Ap^3\end{aligned}\right\}\quad\text{.........(11)}.$$

If only the direction of motion is required, we eliminate v between the equations (4) and (7). Remembering that $\rho=D^2/p$, we see that the direction of motion at any point of the path is parallel to that semi-diameter D whose length is given by

$$\frac{2(U+C)}{D^2}=Ap^2+\frac{N}{p} \quad\text{...................(12)}.$$

Supposing the condition (9) to be satisfied we notice that when the initial velocity and direction of motion are such that the equation (11) gives $A = 0$, it follows by (10) that *the pressure R is zero throughout the motion. The particle is therefore free and moves unconstrained by the ellipsoid.* Conversely, if the particle, when properly projected, can freely describe a curve on the ellipsoid, the condition (9) is satisfied. If it can describe the same curve when otherwise projected, the pressure varies as p^3.

If the components X, Y, Z do not satisfy the condition (9), we may sometimes make them do so by adding to them the components of an arbitrary normal force F and subtracting F from the reaction R. The condition (9) then becomes

$$\frac{1}{a^2x^2}\frac{d}{dt}(Xx^3)+\frac{1}{b^2y^2}\frac{d}{dt}(Yy^3)+\frac{1}{c^2z^2}\frac{d}{dt}(Zz^3)=p^2\frac{d}{dt}\left(\frac{F}{p^3}\right),$$

where F is an arbitrary function of x, y, z and p is a function of x, y, z given by (6). The equation (10) then becomes $R=F+Ap^3$.

It is only necessary that the condition (9) should hold for the path of the particle, but as this is generally unknown, the condition should be true for every arc on the ellipsoid.

570. *Ex.* A particle is acted on by a centre of attractive force situated at the centre of the ellipsoid, the force being κr. If D is the semi-diameter parallel to the tangent to the path, prove that

$$\frac{v}{D}=\surd(\kappa+Ap^2), \qquad v^2=2C-\kappa r^2.$$

These reduce to the ordinary formulæ of central forces when $A=0$.

Since $X=-\kappa x$, &c. the condition (9) is satisfied. The first of the results to be proved then follows from (11), for $N=\kappa p$.

571. *Ex.* A particle P moves on the ellipsoid under the action of a force $Y=-\kappa/y^3$, whose direction is always parallel to the axis of y, and is projected from any point P with a velocity $v^2=\kappa/y^2$ in a direction perpendicular to the geodesic joining P to an umbilicus. Prove that the path is a geodesic circle having the umbilicus for centre, i.e. the geodesic distance of P from the umbilicus is constant*.

We see by substitution that the condition (9) is satisfied by this law of force. The path is therefore given by

$$\frac{v^2}{D^2}=Ap^2+\frac{N}{p}, \qquad v^2=\frac{\kappa}{y^2}+2C,$$

where, as before, D is the semi-diameter parallel to the tangent to the path. Since the cosine of the angle the normal makes with the axis of y is py/b^2, we have

* This result is due to W. R. W. Roberts, who gives a proof by elliptic coordinates in the *Proceedings of the London Math. Soc.* 1883.

$N=\kappa p/b^2y^2$. The conditions of projection show that $C=0$. Hence $\dfrac{1}{D^2}=\dfrac{A}{\kappa}p^2y^2+\dfrac{1}{b^2}$
If ρ, σ are the semi-axes of the diametral plane of P

$$\rho^2+\sigma^2=a^2+b^2+c^2-r^2, \qquad \rho\sigma=abc/p.$$

If also D, D' are two semi-diameters *at right angles* of the same plane

$$\frac{1}{D^2}+\frac{1}{D'^2}=\frac{1}{\rho^2}+\frac{1}{\sigma^2}=\frac{a^2+b^2+c^2-r^2}{a^2b^2c^2}p^2;$$

$$\therefore \frac{1}{p^2D'^2}=\frac{a^2+b^2+c^2-r^2}{a^2b^2c^2}-\frac{1}{b^2p^2}-\frac{Ay^2}{\kappa}.$$

Substituting for p and r their Cartesian values

$$\frac{1}{p^2D'^2}=\frac{1}{a^2c^2}+\frac{a^2+c^2}{a^2b^2c^2}\left(1-\frac{x^2}{a^2}-\frac{z^2}{c^2}\right)-\left(\frac{1}{a^2b^2c^2}+\frac{1}{b^6}+\frac{A}{\kappa}\right)y^2.$$

Using the equation to the surface, this becomes

$$\frac{1}{p^2D'^2}=\frac{1}{a^2c^2}+\left\{\frac{(a^2-b^2)(b^2-c^2)}{a^2b^2c^2}-\frac{Ab^4}{\kappa}\right\}\frac{y^2}{b^4}.$$

Since the particle is projected perpendicularly to the geodesic defined by $pD'=ac$, the coefficient of y^2 must be zero. It then follows that throughout the subsequent motion $pD'=ac$, and the path cuts all the geodesics from the umbilicus at right angles. These geodesics are therefore all of constant length.

Let ω be the angle which the geodesic joining the particle P to an umbilicus U makes with the arc joining the umbilici. If ds be an arc of the orthogonal trajectory of the geodesics, $ds=Pd\omega$, where $P=y/\sin\omega$ (Art. 546). Since $v^2=\kappa/y^2$, it follows that the angular velocity ω' of the geodesic radius vector is given by $\omega'=\dfrac{\sqrt{\kappa}}{y^2}\sin\omega$.

When the ellipsoid reduces to a disc lying in the plane xy, the geodesics become straight lines and the geodesic circle reduces to a Euclidian circle having its centre at H (Art. 576). The theorem is then identical with one given by Newton, viz. that a circle can be described under the action of a force $Y=-\kappa/y^3$.

The motion of a particle in a *geodesic circle under the action of a force*, or tension, *along the geodesic radius* is given in Art. 548, where the result is deduced from Gauss' coordinates.

572. *Ex.* 1. A particle, moving on the ellipsoid, is acted on by a centre of force situated at any given point E. If the force F is such that the condition (9) is satisfied, prove that $F=\mu r/P^3$, where r and P are the distances of the particle from E and from the polar plane of E respectively. Thence show that, if the initial conditions are such that the constant $A=0$, the path is a conic and the velocity at any point is given by $v^2=\rho N$.

To prove this we put $X=G(x-\alpha)$, $Y=G(y-\beta)$, $Z=G(z-\gamma)$, where $G=F/r$ and (α, β, γ) are the coordinates of E. Substituting in the equation (9) and remembering (2) Art. 568, we have an easy differential equation to find G. When $A=0$, the particle moves freely on the ellipsoid under the action of a central force. The path is a plane curve and is therefore a conic. The equation of vis viva fails to give the velocity, but this is determined by (11) Art. 569, when the direction of motion is known.

Ex. 2. A particle moving on a prolate spheroid is acted on by a central force tending to one focus and attracting according to the Newtonian law. Prove that the integrals of the equations of motion are

$$\left(\frac{dr}{dt}\right)^2=\frac{2\mu}{r}-\frac{\mu b^2}{ar^2}+A\,\frac{b^2p^2}{a^2}+C, \qquad v^2=\frac{2\mu}{r}+C,$$

where p is the perpendicular from the centre on the tangent plane, r the distance from the focus, and A, B the constants of integration.

573. *Ex.* 1. A particle under the action of no external forces is projected from an umbilicus of an ellipsoid, prove that the path is one of the geodesics defined by $pD=ac$.

Ex. 2. A particle is projected with a velocity v along the surface of an indefinitely thin ellipsoidal shell bounded by similar ellipsoids. Prove that when it leaves the ellipsoid the perpendicular p from the centre on the tangent plane is given by $MP^2R^2=v^2p^2abc$, where R is the radius vector parallel to the initial direction of motion, P the perpendicular on the initial tangent plane, M the attracting mass and a, b, c the semi-axes of the ellipsoid. [Math. Trip. 1860.]

574. *Ex.* Let the forces be such that $\frac{1}{p^3}(Xd\lambda+Yd\mu+Zd\nu)$ is a perfect differential, say dS, for all displacements on the ellipsoid, where λ, μ, ν are the direction cosines of the normal, i.e. $\lambda=px/a^2$, &c. Prove that

$$R+N=2p^3(S+B), \qquad \frac{v^2}{\rho}=p\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}\right)=2p^3(S+B),$$

where B is the constant of integration.

Divide (8), Art. 569, by p^2 and integrate by parts. The integrals of the equations of motion are then obtained by using (6) and (7), remembering that $\rho=D^2/p$.

575. *In order to include in one form all the different cases of paraboloids, cones, and cylinders*, it may be useful to state the results when the quadric on which the particle moves is written in its most general form $\phi(x, y, z)=0$.

Writing $\frac{4}{p^2}=\phi_x^2+\phi_y^2+\phi_z^2$, where suffixes denote partial differential coefficients, let the forces satisfy the condition

$$\frac{1}{\phi_x^2}\frac{d}{dt}(\phi_x^3X)+\frac{1}{\phi_y^2}\frac{d}{dt}(\phi_y^3Y)+\frac{1}{\phi_z^2}\frac{d}{dt}(\phi_z^3Z)=0 \;\ldots\ldots\ldots\ldots\ldots\;(9),$$

for all displacements on the quadric. We then find that the pressure $R=Ap^3$. The three components x', y', z' of the velocity may be deduced from the equations

$$\phi_xx'+\phi_yy'+\phi_zz'=0\ldots\ldots\ldots\ldots(2), \qquad \tfrac{1}{2}(x'^2+y'^2+z'^2)=U+C\ldots\ldots\ldots\ldots(4),$$

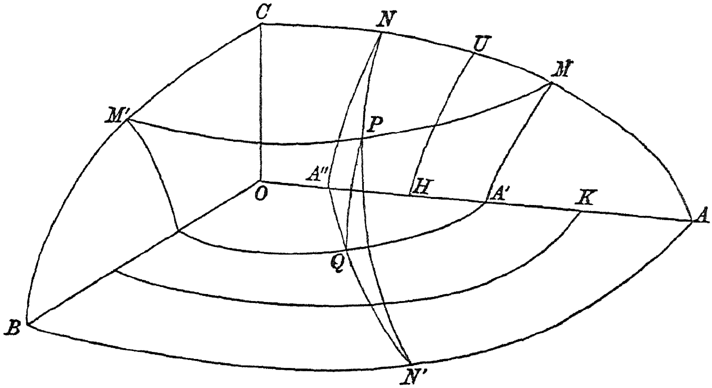
$$\phi_{xx}x'^2+\&c.+2\phi_{xy}x'y'+\&c.+\phi_xX+\phi_yY+\phi_zZ=\frac{2R}{p}\;\ldots\ldots\ldots\ldots\;(5),$$

where the numbers appended to the equations correspond to those in Arts. 568, &c.

576. Elliptic coordinates. *Preliminary statement.* The position of the particle P in space is defined by the intersection of three quadrics confocal to a given quadric. In the figure ABC, $A'MM'$, $A''NN'$ are respectively the ellipsoid, hyperboloid of one sheet and that of two sheets; only that part of each being drawn which lies in the positive octant. Let their major axes $OA=\lambda$, $OA'=\mu$,

$OA''=\nu$. Let a, b, c be the three axes of any confocal. If $a^2-b^2=h^2$, $a^2-c^2=k^2$, then $OH=h$, $OK=k$ are the major axes of the focal conics.

The quantities λ, μ, ν are the elliptic coordinates of P; the first λ is always positive and greater than k; the second μ is less than k and greater than h; the



third ν is less than h, and changes sign when the particle crosses the plane of yz. The y axes of the quadrics are $\sqrt{(\lambda^2-h^2)}$, $\sqrt{(\mu^2-h^2)}$, $\sqrt{(\nu^2-h^2)}$; two of these are real and the third is imaginary. These radicals are positive when the particle lies in the positive octant, but the second or third vanishes and changes sign when the particle crosses the plane of xz, according as it travels along PN or PM. Similar remarks apply to the z axes.

The major axes of the three confocals which intersect in any point (x, y, z) are given by the cubic

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-h^2}+\frac{z^2}{a^2-k^2}=1,$$

where h and k are the constants of the system. Clearing of fractions and arranging the cubic in descending powers of a^2, we see that the three roots λ^2, μ^2, ν^2 are such that

$$\left.\begin{array}{c}\lambda^2+\mu^2+\nu^2=x^2+y^2+z^2+h^2+k^2\\ \lambda^2\mu^2+\mu^2\nu^2+\nu^2\lambda^2=h^2(x^2+z^2)+k^2(x^2+y^2)+h^2k^2\\ \lambda\mu\nu=hkx\end{array}\right\}\ldots\ldots\ldots\ldots\ldots(1).$$

From the third equation we infer by symmetry

$$\left.\begin{array}{c}\sqrt{(\lambda^2-h^2)}\sqrt{(\mu^2-h^2)}\sqrt{(h^2-\nu^2)}=h\sqrt{(k^2-h^2)}\,y\\ \sqrt{(\lambda^2-k^2)}\sqrt{(k^2-\mu^2)}\sqrt{(k^2-\nu^2)}=k\sqrt{(k^2-h^2)}\,z\end{array}\right\}\ldots\ldots\ldots\ldots\ldots(2).$$

577. *To prove that the velocity v of a particle in elliptic coordinates is given by*

$$v^2=\frac{(\lambda^2-\mu^2)(\lambda^2-\nu^2)\lambda'^2}{(\lambda^2-h^2)(\lambda^2-k^2)}+\frac{(\mu^2-\lambda^2)(\mu^2-\nu^2)\mu'^2}{(\mu^2-h^2)(\mu^2-k^2)}+\frac{(\nu^2-\lambda^2)(\nu^2-\mu^2)\nu'^2}{(\nu^2-h^2)(\nu^2-k^2)}\ \ldots\ (3).$$

We notice that the three quadrics confocal to a given quadric cut each other at right angles at P, so that the square of the velocity

is the sum of the squares of the normal components of velocity. It is therefore sufficient to prove that the first term is the square of the component normal to the ellipsoid, the other terms following by symmetry. If p is the perpendicular on the tangent plane to the ellipsoid, the normal component is p'. Let (l, m, n) be the direction cosines of p, then

$$p^2 = \lambda^2 l^2 + (\lambda^2 - h^2)\, m^2 + (\lambda^2 - k^2)\, n^2$$
$$= \lambda^2 - h^2 m^2 - k^2 n^2; \quad \therefore\ pp' = \lambda\lambda'.$$

If D_1, D_2 are the semi-diameters of the ellipsoid respectively normal to the tangent planes at P to the two hyperboloids, we know that

$$D_1^2 = \lambda^2 - \mu^2, \quad D_2^2 = \lambda^2 - \nu^2, \qquad p^2 = \frac{\lambda^2 (\lambda^2 - h^2)(\lambda^2 - k^2)}{D_1^2 D_2^2};$$
$$\therefore\ p'^2 = \frac{(\lambda^2 - \mu^2)(\lambda^2 - \nu^2)\, \lambda'^2}{(\lambda^2 - h^2)(\lambda^2 - k^2)}.$$

See also Salmon's *Solid Geometry*, Art. 410.

578. *To find the motion of a particle on an ellipsoid in elliptic coordinates.* Let the ellipsoid on which the particle moves be defined by a given value of λ. The mass being taken as unity the vis viva is determined by

$$2T = v^2 = (\mu^2 - \nu^2) \left\{ \frac{(\mu^2 - \lambda^2)\, \mu'^2}{(\mu^2 - h^2)(\mu^2 - k^2)} - \frac{(\nu^2 - \lambda^2)\, \nu'^2}{(\nu^2 - h^2)(\nu^2 - k^2)} \right\} \ldots(4).$$

This we write for brevity in the form

$$2T = M \{P\mu'^2 + Q\nu'^2\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

If we express the work function U in terms of (λ, μ, ν), we have (since λ is constant) the Lagrangian function $T + U$ expressed in terms of two independent coordinates μ, ν.

Comparing (5) with Liouville's form, Art. 522, we may obviously solve the Lagrangian equations by proceeding as in that article. The results are that when the forces are such that the work function takes the form

$$(\mu^2 - \nu^2)\, U = F_1(\mu) + F_2(\nu) \ldots\ldots\ldots\ldots\ldots\ldots(A),$$

the integrals are

$$\left.\begin{aligned} \tfrac{1}{2}(\mu^2 - \nu^2)^2 \frac{(\mu^2 - \lambda^2)\, \mu'^2}{(\mu^2 - h^2)(\mu^2 - k^2)} &= F_1(\mu) + C\mu^2 + A \\ -\tfrac{1}{2}(\mu^2 - \nu^2)^2 \frac{(\nu^2 - \lambda^2)\, \nu'^2}{(\nu^2 - h^2)(\nu^2 - k^2)} &= F_2(\nu) - C\nu^2 - A \end{aligned}\right\} \ldots(B).$$

There is also the equation of vis viva

$$\tfrac{1}{2}v^2 = U + C \text{.........................(C)}.$$

Dividing one of the equations (B) by the other, and remembering that λ is constant, the equation of the path takes the forms

$$\frac{(\mu^2-\lambda^2)(d\mu)^2}{(\mu^2-h^2)(\mu^2-k^2)\{F_1(\mu)+C\mu^2+A\}} = \frac{-(\nu^2-\lambda^2)(d\nu)^2}{(\nu^2-h^2)(\nu^2-k^2)\{F_2(\nu)-C\nu^2-A\}} \text{ ...(D)},$$

in which the variables are separated.

579. *Ex.* 1. Let v_1 and v_2 be the components of the velocity of the particle in the directions of the lines of curvature defined by μ=constant and ν=constant respectively. Prove that

$$\frac{1}{2}v_1^2 = \frac{F_2(\nu)-C\nu^2-A}{\mu^2-\nu^2}, \qquad \frac{1}{2}v_2^2 = \frac{F_1(\mu)+C\mu^2+A}{\mu^2-\nu^2}.$$

Prove also that the pressure R on the particle is given by

$$R+N = \left\{\frac{F_1(\mu)+C\mu^2+A}{\lambda^2-\mu^2} + \frac{F_2(\nu)-C\nu^2-A}{\lambda^2-\nu^2}\right\}\frac{2p}{\mu^2-\nu^2},$$

where p is the perpendicular on the tangent plane and N the normal impressed force. The value of p in elliptic coordinates is given in Art. 577. See Art. 568.

Ex. 2. Supposing that the equation (D) of Art. 578 is written in the form $P\,d\mu = Q\,d\nu$ in which the variables are separated, show that the time

$$t = \int P\mu^2\,d\mu - \int Q\nu^2\,d\nu. \qquad \text{[\textit{Liouville}, XI.]}$$

The equations (B) become

$$(\mu^2-\nu^2)\,P\,d\mu = dt, \qquad (\mu^2-\nu^2)\,Q\,d\nu = dt.$$

Multiplying these by μ^2, ν^2 respectively and subtracting we obtain the result.

580. *To translate the elliptic expressions into Cartesian geometry* we use the equations (1) and (2) of Art. 576. Let the normals at the four umbilici U_1, U_2, &c. intersect the major axis in the two points E_1, E_2, which of course are equally distant from the centre O. We easily find that

$$OE_1 = \frac{hk}{\lambda}, \qquad E_1U_1 = \frac{bc}{a} = \frac{\sqrt{(\lambda^2-h^2)}\sqrt{(\lambda^2-k^2)}}{\lambda} \text{ .... (1)}.$$

The equations (1) Art. 576 give

$$(\mu \pm \nu)^2 = \left(x \pm \frac{hk}{\lambda}\right)^2 + y^2 + z^2 - \frac{(\lambda^2-h^2)(\lambda^2-k^2)}{\lambda^2}.$$

Let r_1, r_2 be the distances of the particle from the points E_1, E_2, and let m be the distance of E_1 from the umbilicus U_1; then

$$(\mu-\nu)^2 = r_1^2 - m^2, \qquad (\mu+\nu)^2 = r_2^2 - m^2 \text{........................(2)}.$$

From these μ, ν may be found in terms of x, y, z and the constant λ.

581. *Ex.* Show that the equation $U(\mu^2-\nu^2) = F_1(\mu)+F_2(\nu)$ is equivalent to $\dfrac{d^2}{d\rho_1^2}(U\rho_1\rho_2) = \dfrac{d^2}{d\rho_2^2}(U\rho_1\rho_2)$, where $\rho_1 = \sqrt{(r_1^2-m^2)}$, $\rho_2 = \sqrt{(r_2^2-m^2)}$.

We have $\dfrac{d^2}{d\mu\,d\nu}U(\mu^2-\nu^2) = 0$, and by (2) Art. 580

$$\frac{d}{d\mu} = \frac{d}{d\rho_2} + \frac{d}{d\rho_1}, \qquad \frac{d}{d\nu} = \frac{d}{d\rho_2} - \frac{d}{d\rho_1}.$$

The result follows at once.

582. *The condition* (A) *of Art.* 578, *viz.*

$$(\mu^2-\nu^2)\,U=F_1(\mu)+F_2(\nu) \quad\text{.............................. (A)},$$

can be satisfied by several laws of force.

1. Let the force tend to the centre of the ellipsoid and vary as the distance. Representing the force by Hr, we have, by (1) Art. 576,

$$U=-\tfrac{1}{2}Hr^2=-\tfrac{1}{2}H\{\mu^2+\nu^2+(\lambda^2-h^2-k^2)\};$$

$$\therefore\ F_1(\mu)=-\tfrac{1}{2}H\{\mu^4+(\lambda^2-h^2-k^2)\mu^2\},\qquad F_2(\nu)=\tfrac{1}{2}H\{\nu^4+(\lambda^2-h^2-k^2)\nu^2\}.$$

Substituting these in the equations (B), the motion is known.

2. Let the direction of the force be parallel to the axis of x, and $X=-2H/x^3$. Then

$$U=\frac{H}{x^2}=\frac{Hh^2k^2}{\lambda^2\mu^2\nu^2},\qquad \therefore\ (\mu^2-\nu^2)\,U=\frac{Hh^2k^2}{\lambda^2}\left\{-\frac{1}{\mu^2}+\frac{1}{\nu^2}\right\}.$$

3. Let the work function $U=\dfrac{H}{\sqrt{(r_1^2-m^2)}}$, where r_1 is the distance of the particle from the point E_1, Art. 580. We then have

$$U=\frac{H}{\mu-\nu},\qquad \therefore\ (\mu^2-\nu^2)\,U=H(\mu+\nu).$$

To find the force we notice that since $dU/d\lambda=0$, the direction of the force is tangential to the ellipsoid. Also

$$(\mu-\nu)^2=x^2+y^2+z^2-2xhk/\lambda-\lambda^2+h^2+k^2;$$

$$\therefore\ X=\frac{dU}{dx}=-\frac{H}{(\mu-\nu)^3}\left\{x-\frac{hk}{\lambda}+\left(\frac{hkx}{\lambda^2}-\lambda\right)\frac{d\lambda}{dx}\right\},$$

with similar expressions for Y and Z. Now the equation to the ellipsoid being $\lambda=$ constant, the last term of each of the three expressions represents the component of a normal force. This normal force has no effect on the motion. *Taking only the remaining terms we see that X, Y, Z are the components of a central force tending to the point E whose magnitude is* $\dfrac{Hr_1}{(r_1^2-m^2)^{\frac{3}{2}}}$. When the ellipsoid is reduced to a disc, $\lambda=k$ (Art. 576), and $m=0$ (Art. 580). The point E_1 becomes a focus and the law of force is the inverse square.

583. *Ex.* 1. Show that a particle can describe the line of curvature defined by $\mu=\mu_0$ under the action of the central force $\dfrac{Hr_1}{(r_1^2-m^2)^{\frac{3}{2}}}$ tending to the point E_1. Show also that the velocity at any point is then given by $v^2=H\left\{\dfrac{2}{(r_1^2-m^2)^{\frac{1}{2}}}-\dfrac{1}{\mu_0}\right\}$. We notice that when the ellipsoid reduces to a plane, $m=0$, and this becomes the common expression for the velocity under the action of a central force varying as the inverse square.

Referring to the general expressions marked (A) and (B) in Art. 578, we see that the particle will describe the line of curvature if both $\mu'=0$ and $\mu''=0$ when $\mu=\mu_0$. This will be the case if we choose the constants C and A so that

$$F_1(\mu)+C\mu^2+A=(\mu-\mu_0)^2\,\phi(\mu),$$

where $\phi(\mu)$ is some function of μ. Supposing this done, we have, when $\mu=\mu_0$,

(Art. 579) $$\tfrac{1}{2}v^2=U+C=\frac{F_2(\nu)-C\nu^2-A}{\mu_0^2-\nu^2}.$$

In the special case proposed $U=H/(\mu-\nu)$. We have therefore to make $C\mu^2+H\mu+A=(\mu-\mu_0)^2 C$. This gives $-2C\mu_0=H$, $A=C\mu_0^2$. Also $F_2(\nu)=H\nu$.

$$\therefore\ v^2=H\left\{\frac{2}{\mu_0-\nu}-\frac{1}{\mu_0}\right\}=H\left\{\frac{2}{(r_1^2-m^2)^{\frac{1}{2}}}-\frac{1}{\mu_0}\right\}.$$

Ex. 2. A particle is constrained to move on the surface $y=x\tan nz$. By putting $x=\mu\cos nz$, $y=\mu\sin nz$, we have

$$v^2=(\mu^2n^2+1)\left\{\frac{\mu'^2}{\mu^2n^2+1}+z'^2\right\}=M(\xi'^2+z'^2).$$

Hence show that when the forces are such that

$$(\mu^2n^2+1)\,U=F_1(\mu)+F_2(z),$$

the Lagrangian equations can be integrated. The path is given by

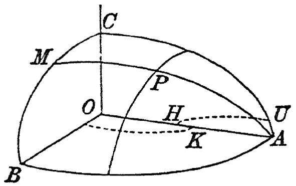
$$\frac{\mu'^2}{(\mu^2n^2+1)\{F_1(\mu)+C(\mu^2n^2+1)+A\}}=\frac{z'^2}{F_2(z)-A}\qquad\text{[Liouville, 1846.]}$$

If the particle is acted on by a force tending directly from the axis of z and varying as the distance from that axis, find the components of velocity along the lines of curvature.

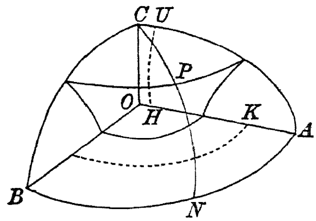
584. Spheroids. When the ellipsoid on which the particle moves becomes a spheroid either prolate or oblate, the formulæ (A) and (B) of Art. 578 require some slight modifications.

Let (λ, b, c), (μ, b', c'), (ν, b'', c'') be the semi-axes of the three quadrics which intersect in P; then also $a=\lambda$, $a'=\mu$, $a''=\nu$.

In a prolate spheroid $b=c$, $h=k$, and the focal conics become coincident with



Prolate.



Oblate.

OH and HA. The axes of the hyperboloid of one sheet are $\mu=h$, $b'=0$, $c'=0$; it therefore reduces to the two planes $y^2/b'^2+z^2/c'^2=0$, the ratio b'/c' being indeterminate. Art. 576.

In an oblate spheroid $\lambda=b$, $h=0$; one focal conic becomes coincident with OC, while the other is a circle of radius k. The axes of the hyperboloid of two sheets are $\nu=0$, $b''=0$, $c''^2=-k^2$; it therefore reduces to the two planes $x^2/\nu^2+y^2/b''^2=0$, the limiting ratio ν/b'' being indeterminate.

In the figure the positions of the focal conics *just before* they assume their limiting positions are represented by the dotted lines, while PM or PN represents one of the planes assumed by the hyperboloid.

Before taking the limits of the equations (A) and (B) we shall make a change of variables. In the prolate spheroid we replace μ by a new variable ϕ, such that

$$\tan^2\phi=-\frac{\mu^2-k^2}{\mu^2-h^2},\qquad \therefore\ \sin^2\phi=-\frac{\mu^2-k^2}{k^2-h^2},\qquad \therefore\ -\phi'^2=\frac{\mu^2\mu'^2}{(\mu^2-h^2)(\mu^2-k^2}$$

Thus $\tan\phi$ varies between the limits 0 and ∞ as μ varies between k and h. Since $b'^2=\mu^2-h^2$, $c'^2=\mu^2-k^2$, and $y^2/b'^2+z^2/c'^2=0$, it is clear that ϕ is ultimately the angle the plane PM makes with the plane AB. Putting $\mu=h$, the formulæ (A) and (B) become

$$(h^2-\nu^2)\,U=f_1(\phi)+F_2(\nu),$$

$$-\tfrac{1}{2}(h^2-\nu^2)^2\,\frac{(h^2-\lambda^2)\,\phi'^2}{h^2}=f_1(\phi)+Ch^2+A,$$

$$-\tfrac{1}{2}(\nu^2-\lambda^2)\,\nu'^2=F_2(\nu)-C\nu^2-A.$$

In the oblate spheroid, we replace ν by the variable ϕ where

$$\tan^2\phi=-\frac{\nu^2-h^2}{\nu^2},\qquad \therefore\ \nu=h\cos\phi,\qquad \therefore\ -\phi'^2=\frac{\nu'^2}{\nu^2-h^2},$$

thus $\tan\phi$ varies between 0 and ∞ as ν varies between h and 0. Also since $x^2/\nu^2+y^2/b''^2=0$, ϕ is ultimately the angle the plane PM makes with the plane AC. Putting $\nu=0$, $h=0$, the limiting forms of the equations (A), (B) are

$$\mu^2U=F_1(\mu)+f_2(\phi),$$

$$\tfrac{1}{2}\mu^2\frac{(\mu^2-\lambda^2)\,\mu'^2}{\mu^2-k^2}=F_1(\mu)+C\mu^2+A,$$

$$\tfrac{1}{2}\mu^4\frac{\lambda^2\phi'^2}{k^2}=f_2(\phi)-A.$$

CHAPTER VIII.

SOME SPECIAL PROBLEMS.

Motion under two centres of force.

585. *To find the motion of a particle of unit mass in on plane under the action of two centres of force*.*

Let the position of a point P be defined as the intersection of two confocal conics, the foci being H_1, H_2, and let $OH_1 = h$. Let the semi-major axes be $OA = \mu$, $OA' = \nu$: the semi-minor axes are therefore $\sqrt{(\mu^2 - h^2)}$, $\sqrt{(\nu^2 - h^2)}$.

Since $\dfrac{x^2}{\mu^2} + \dfrac{y^2}{\mu^2 - h^2} = 1$, we have

$$\mu^4 - (x^2 + y^2 + h^2)\,\mu^2 + h^2x^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots(1).$$

The relations between the elliptic coordinates μ, ν of any point P and the Cartesian coordinates x, y are therefore

$$x = \frac{\mu\nu}{h}, \quad y = \frac{(\mu^2 - h^2)^{\frac{1}{2}}\,(\nu^2 - h^2)^{\frac{1}{2}}}{h\sqrt{-1}}, \quad r^2 = \mu^2 + \nu^2 - h^2,$$

where r is the distance from the centre. We also have $r_2 = \mu + \nu$, $r_1 = \mu - \nu$, where r_1, r_2 are the distances of P from the foci.

* Euler was the first who attacked the problem of the motion of a particle in one plane about two fixed centres of force, *Mémoires de l'Académie de Berlin*, 1760. Lagrange, in the *Mécanique Analytique*, page 93, begins by excusing himself for attempting a problem which has nothing corresponding to it in the system of the world, where all the centres of force are in motion. He supposes the motion to be in three dimensions and obtains a solution where the forces are $\alpha/r^2 + 2\gamma r$ and $\beta/r^2 + 2\gamma r$. Legendre in his *Fonctions elliptiques* pointed out that the variables used by Euler were really elliptic coordinates, and Serret remarks that this is the first time these coordinates were used. Jacobi took this problem as an example of his principle of the least multiplier, *Crelle*, XXVII. and XXIX. Liouville in 1846 and 1847 gives two methods of solution, the first by Lagrange's equations and the second by the Hamiltonian equations. Serret extends Liouville's first method to three dimensions, *Liouville's Journal*, XIII. 1848, and gives a history of the problem. Liouville in the same volume gives a further communication on the subject.

Proceeding as in Art. 577, the velocity v of the particle expressed in elliptic coordinates is

$$2T = v^2 = (\mu^2 - \nu^2)\left\{\frac{\mu'^2}{\mu^2 - h^2} - \frac{\nu'^2}{\nu^2 - h^2}\right\} \text{ ....(2)},$$

where the accent represents d/dt. Comparing this with Liouville's form

$$2T = M(P\mu'^2 + Q\nu'^2)$$

in Art. 522, we may obviously solve the Lagrangian equations by proceeding as in that Article. The results are that when the work function has the form

$$(\mu^2 - \nu^2)U = F_1(\mu) + F_2(\nu) \text{ ....(3)},$$

we have the two integrals

$$\left.\begin{aligned} \tfrac{1}{2}(\mu^2 - \nu^2)^2 \frac{\mu'^2}{\mu^2 - h^2} &= F_1(\mu) + C\mu^2 + A \\ -\tfrac{1}{2}(\mu^2 - \nu^2)^2 \frac{\nu'^2}{\nu^2 - h^2} &= F_2(\nu) - C\nu^2 - A \end{aligned}\right\} \text{ ....(4)}.$$

There is also the equation of vis viva which may be deduced from these by simple addition, viz.

$$\tfrac{1}{2}v^2 = U + C \text{ ....(5)}.$$

586. Let the central forces tending to the foci be respectively H_1/r_1^2 and H_2/r_2^2. We then have

$$U = \frac{H_1}{r_1} + \frac{H_2}{r_2}; \quad \therefore (\mu^2 - \nu^2)U = (H_1 + H_2)\mu + (H_1 - H_2)\nu \text{ ...(6)}.$$

The integrals (4) then become

$$\left.\begin{aligned} \tfrac{1}{2}(\mu^2 - \nu^2)^2 \frac{\mu'^2}{\mu^2 - h^2} &= K_1\mu + C\mu^2 + A \\ -\tfrac{1}{2}(\mu^2 - \nu^2)^2 \frac{\nu'^2}{\nu^2 - h^2} &= K_2\nu - C\nu^2 - A \end{aligned}\right\} \text{ ....(7)},$$

where $K_1 = H_1 + H_2$, $K_2 = H_1 - H_2$. To find the path we eliminate t,

$$\frac{(d\mu)^2}{(\mu^2 - h^2)(C\mu^2 + K_1\mu + A)} = \frac{-(d\nu)^2}{(\nu^2 - h^2)(-C\nu^2 + K_2\nu - A)} = \frac{2(dt)^2}{(\mu^2 - \nu^2)^2} \text{ ...... (8)}.$$

The initial values of μ, μ', ν, ν' being given, the equations (7) determine the constants A, C. Another constant is introduced by the integration of (8) which is also determined by the initial values of μ, ν. A fourth constant makes its appearance when the time is found in terms of either μ or ν.

587. *Ex.* 1. Show that the particle will describe the ellipse defined by $\mu=\mu_0$, if the particle is projected along the tangent at any point with a velocity v given by

$$v^2=H_1\left(\frac{2}{r_1}-\frac{1}{\mu_0}\right)+H_2\left(\frac{2}{r_2}-\frac{1}{\mu_0}\right).$$

To prove this we notice that if the particle describe the ellipse, μ is constant throughout the motion, and the values of μ', μ'' given by (7) must be zero. The right-hand side of that equation must take the form $C(\mu-\mu_0)^2$, and therefore $-2C\mu_0=K_1$. Substituting for C in the equation of vis viva (5) the result follows at once. See also Art. 274.

Ex. 2. A particle is projected so that both the constants A and C are zero. Show that the velocity is that due to an infinite distance and that the path is given by

$$\int\frac{d\phi}{\surd(1-\frac{1}{2}\sin^2\phi)}=\left(\frac{K_1}{K_2}\right)^{\frac{1}{2}}\int\frac{d\theta}{\surd(1-\frac{1}{2}\sin^2\theta)}+B,$$

where $\mu=h\sec^2\phi$, $\nu=h\cos^2\theta$ and B is a constant.

Ex. 3. A particle moves under the action of two equal centres of force, one attracting and the other repelling like the poles of a magnet. The particle is projected with a velocity due to an infinite distance. Show that if the direction of projection be properly chosen the particle will oscillate in a semi-ellipse, the two poles being the foci. If otherwise projected the path is given by

$$\sqrt{\frac{h}{\beta}}\log\{\mu+\surd(\mu^2-h^2)\}+B=\int\frac{d\phi}{\surd(1-k\sin^2\phi)},$$

where $\nu=h\cos^2\phi+\beta\sin^2\phi$, $2k=1-\beta/h$ and $A=2H\beta$.

Ex. 4. Prove that the lemniscate, $rr'=c^2$, can be described under the action of two centres of force each H/r^3 tending to the foci, provided the velocity at the node is $\frac{2}{c}\sqrt{\frac{2H}{3}}$. See Art. 190, Ex. 11.

588. *To find the motion of a particle of unit mass in three dimensions under the action of two centres of force attracting according to the Newtonian law.*

Let the two centres of force H_1, H_2, be situated in the axis of z and let the origin O bisect the distance H_1H_2. Let ϕ be the angle the plane zOP makes with zOx and let ρ be the distance of P from Oz.

Since the impressed forces have no moment about Oz, we have by the principle of angular momentum (Art. 492),

$$\rho^2\phi'=B \qquad\qquad (1).$$

We now adopt the method explained in Art. 495. We treat the particle as if it were moving in a fixed plane zOP under the influence of the two centres of force together with an additional force $\rho\phi'^2=B^2/\rho^3$ tending from the axis of z. This problem has been partly solved in Art. 585; it only remains to consider the

effect of the additional force. This force adds the term $-B^2/2\rho^2$ to the work function U.

Taking H_1, H_2 as the foci of a system of confocal conics, let μ, ν be the elliptic coordinates of P. As before, we suppose that the work function U of the impressed forces satisfies the condition

$$(\mu^2 - \nu^2)\, U = F_1(\mu) + F_2(\nu) \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Since ρ is the ordinate of the conics [Art. 585],

$$\rho^2 = \frac{(\mu^2 - h^2)(\nu^2 - h^2)}{-h^2}; \qquad \therefore\ \frac{\mu^2 - \nu^2}{\rho^2} = \frac{h^2}{\mu^2 - h^2} - \frac{h^2}{\nu^2 - h^2} \ldots(3).$$

The term to be added to U has therefore the same form as those already existing in U and shown in (2). To obtain the integrals we have merely to add the terms given in (3), (after multiplication by $-\frac{1}{2}B^2$) to the functions F_1, F_2.

In this way, we find the integrals

$$\left.\begin{aligned} \tfrac{1}{2}(\mu^2 - \nu^2)^2 \frac{\mu'^2}{\mu^2 - h^2} &= F_1(\mu) + C\mu^2 - \tfrac{1}{2}\frac{B^2h^2}{\mu^2 - h^2} + A \\ -\tfrac{1}{2}(\mu^2 - \nu^2)^2 \frac{\nu'^2}{\nu^2 - h^2} &= F_2(\nu) - C\nu^2 + \tfrac{1}{2}\frac{B^2h^2}{\nu^2 - h^2} - A \end{aligned}\right\} \ldots(4).$$

When the central forces follow the Newtonian law,

$$U = \frac{H_1}{r_1} + \frac{H_2}{r_2}; \qquad \therefore\ (\mu^2 - \nu^2)\, U = K_1\mu + K_2\nu,$$

where $K_1 = H_1 + H_2$, $K_2 = H_1 - H_2$, as in Art. 586. We therefore write in the solution (4), $F_1(\mu) = K_1\mu$, $F_2(\nu) \mp K_2\nu$.

If the particle is acted on by a third centre of force situated at the origin and attracting as the distance, we add to the expression for U the term $-\frac{1}{2}H_3r^2 = -\frac{1}{2}H_3(\mu^2 + \nu^2 - h^2)$. The effect of this is to increase the functions F_1, F_2 by $-\frac{1}{2}H_3(\mu^4 - h^2\mu^2)$, and $\frac{1}{2}H_3(\nu^4 - h^2\nu^2)$ respectively.

In the same way if the particle is also acted on by a force tending directly from the axis of z and equal to κ/ρ^3, or a force parallel to z and equal to κ/z^3, the effect is merely to give additional terms to the functions F_1 and F_2. See Art. 582.

589. *Ex.* A particle P moves under the attraction of two centres of force at A and B. If the angles PAB, PBA be respectively θ_1, θ_2, the distances AP, BP be r_1, r_2, and the accelerations be μ_1/r_1^2, μ_2/r_2^2, prove that

$$\left(r_1^2 \frac{d\theta_1}{dt}\right)\left(r_2^2 \frac{d\theta_2}{dt}\right) = a(\mu_1 \cos\theta_1 + \mu_2 \cos\theta_2) + C,$$

where $AB = a$, C is a constant and the motion is in one plane.

If the motion is in three dimensions, prove that

$$\left(r_1^2 \frac{d\theta_1}{dt}\right)\left(r_2^2 \frac{d\theta_2}{dt}\right) + h^2 \cot\theta_1 \cot\theta_2 = a\,(\mu_1 \cos\theta_1 + \mu_2 \cos\theta_2) + C,$$

where h is the areal description round the line of centres. [Coll. Ex. 1895.]

On Brachistochrones.

590. Preliminary Statement. Let a particle P, projected from a point A at a time t_0 with a velocity v_0, move along a smooth fixed wire under the influence of forces whose potential U is a given function of the coordinates of P, and let the particle arrive at a point B at a time t_1 with a velocity v_1. Let us suppose that the circumstances of the motion are slightly varied. Let a particle start from a neighbouring point A' at a time $t_0 + \delta t_0$ with a velocity $v_0 + \delta v_0$. Let it be constrained by a smooth wire to describe an arbitrary path nearly coincident with the former under forces whose potential is the same function of the coordinates as before, and let it arrive at a point B' near the point B at a time $t_1 + \delta t_1$ with velocity $v_1 + \delta v_1$.

According to the same notation, if x, y, z; x', y', z', are the coordinates and resolved velocities at any point P of the first path at the time t, then $x + \delta x$, &c.; $x' + \delta x'$, &c., are the coordinates and resolved velocities at any point P' of the varied path occupied by the particle at the time $t + \delta t$.

Let P, Q be any two points on the two paths simultaneously occupied at the time t. Let the coordinates of Q be $x + \Delta x$, $y + \Delta y$, &c. Then δx exceeds Δx by the space described in the time δt,

$$\therefore \Delta x = \delta x - (x' + \delta x')\,\delta t = \delta x - x'\delta t$$

when quantities of the second order are neglected.

We may regard δx, δy, δz, as any indefinitely small arbitrary functions of x, y, z, limited only by the geometrical conditions of the problem.

We here consider two independent changes of the coordinates. There are (1) the differentials dx, dy, dz when the particle travels along the undisturbed path, and (2) the variations δx, δy, δz when the particle is displaced to some neighbouring path. It follows from the independence of these two displacements that $d\delta x = \delta dx$.

591. The Brachistochrone. *A particle of unit mass moves under the action of forces so that its velocity v at any point is given by $\frac{1}{2}v^2 = U + C$, where U is a known function of the coordinates, the constant C being also known. Supposing the initial and final positions A, B to lie on two given surfaces, it is required to find the path the particle must be constrained to take that the time of transit may be a minimum*.*

* An account of the early history of this problem is given in Ball's *Short History of Mathematics*. Passing to later times, the theorem $v = Ap$ for a central force is given by Euler, *Mechanica*, vol. II. There is a memoir by Roger in *Liouville's Journal*, vol. XIII. 1848; he discusses the brachistochrone on a surface

The time t of transit being $t = \int ds/v$, we have to make this integral a minimum. Since a variation is only a kind of differential, we follow the rules of the differential calculus and make the first variation of t equal to zero. Let the curve AB be varied into a neighbouring curve $A'B'$, each element being varied into a corresponding element. Since the number of elements is not altered, the variation of the integral is the integral of the variation. Writing ϕ for $1/v$ to avoid fractions, we have

$$\delta t = \int \delta(\phi\, ds) = \int (\phi\, d\delta s + ds\,\delta\phi).$$

Since $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$, we have

$$ds\,\delta ds = dx\,\delta dx + dy\,\delta dy + dz\,\delta dz;$$

$$\therefore\ \delta t = \int \phi \left(\frac{dx}{ds}\, d\delta x + \frac{dy}{ds}\, d\delta y + \&c.\right) + \int \left(\frac{d\phi}{dx}\,\delta x + \frac{d\phi}{dy}\,\delta y + \&c.\right) ds.$$

Integrating the first three terms by parts,

$$\delta t = \phi \left(\frac{dx}{ds}\,\delta x + \&c.\right) + \int \left\{\left[\frac{d\phi}{dx} - \frac{d}{ds}\left(\phi\,\frac{dx}{ds}\right)\right]\delta x + \&c.\right\} ds,$$

where the part outside the sign of integration is to be taken between the limits A to B.

We notice that in this variation, C has not been varied. If C were different for the different trajectories, we should have

$$\delta\phi = \frac{d\phi}{dx}\,\delta x + \frac{d\phi}{dy}\,\delta y + \frac{d\phi}{dz}\,\delta z + \frac{d\phi}{dC}\,\delta C.$$

There would then be an additional term inside the integral. It follows that *v^2 is regarded as the same function of x, y, z for all the trajectories.*

Since the time t is to be a minimum for all variations consistent with the given conditions, it must be a minimum when the ends A, B are fixed (Art. 144). We then have at these points $\delta x = 0$, $\delta y = 0$, $\delta z = 0$, and the part outside the integral vanishes.

The required curve must therefore be such that the integral is zero whatever small values the arbitrary functions δx, δy, δz may have. It is proved in the calculus of variations (and is

and generalises Euler's theorem that the normal force is equal to the centrifugal force. Jellett in his *Calculus of Variations*, 1850, proves these theorems and deduces from the principle of least action that the brachistochrone becomes a free path when $v = k^2/v'$. Tait has applied Hamilton's characteristic function to the problem in the *Edinburgh Transactions*, vol. xxiv. 1865, and deduces from a more general theorem the above relation to free motion. Townsend in the *Quarterly Journal*, vol. xiv. 1877, obtains the relation $v = v'$ in free motion, and gives numerous examples. There are also some theorems by Larmor in the *Proceedings of the London Mathematical Society*, 1884.

perhaps evident) that the coefficients of δx, δy, δz must separately vanish. We therefore have, writing $1/v$ for ϕ,

$$\frac{d}{dx}\left(\frac{1}{v}\right)=\frac{d}{ds}\left(\frac{1}{v}\frac{dx}{ds}\right),\quad \frac{d}{dy}\left(\frac{1}{v}\right)=\frac{d}{ds}\left(\frac{1}{v}\frac{dy}{ds}\right),\quad \frac{d}{dz}\left(\frac{1}{v}\right)=\frac{d}{ds}\left(\frac{1}{v}\frac{dz}{ds}\right).$$

These are the differential equations of the brachistochrone.

These three equations really amount to only two, for if we multiply them by $\phi\, dx/ds$, $\phi\, dy/ds$, &c. and add the products, we find

$$\phi\frac{d\phi}{ds}=\frac{1}{2}\frac{d}{ds}\left\{\phi^2\left(\frac{dx}{ds}\right)^2+\phi^2\left(\frac{dy}{ds}\right)^2+\phi^2\left(\frac{dz}{ds}\right)^2\right\},$$

which is an evident identity.

592. Supposing these differential equations to have been solved, it remains to determine the constants of integration. To effect this we resume the expression for δt, now reduced to the part outside the integral sign. We have

$$\delta t=\phi\left(\frac{dx}{ds}\delta x+\frac{dy}{ds}\delta y+\frac{dz}{ds}\delta z\right),$$

which is to be taken between the limits A to B. Since we may vary the ends A, B of the curve, one at a time, along the bounding surface (Art. 144), this expression for δt must be zero at each end. The variations δx, δy, δz are proportional to the direction cosines of the displacement of the end, and dx/ds, &c. are the direction cosines of the tangent to the brachistochrone. This equation therefore implies that *the brachistochrone meets the bounding surface at right angles.*

The expression for δt may be put into a geometrical form which is sometimes useful. Let $\delta\sigma_1$, $\delta\sigma_2$ be the displacements AA', BB' of the two ends. Let θ_1, θ_2 be the angles these displacements respectively make with the tangents at A and B to the brachistochrone AB. Let v_1, v_2 be the velocities at A, B. Then

$$\delta t=\frac{\delta\sigma_2\cos\theta_2}{v_2}-\frac{\delta\sigma_1\cos\theta_1}{v_1}.$$

593. In some problems the velocity v is a given function of the coordinates of one or both ends of the curve. This does not affect the differential equations, for in these the coordinates of the ends, when fixed, are merely constants.

The case is different when we vary the ends in that portion of the expression for δt which is outside the integral sign. We

must add to that expression the terms of $\delta\phi$ due to the variation of the ends. If x_0, y_0, z_0; x_1, y_1, z_1, are the coordinates of the ends A, B, we then have

$$\delta t = \left[\phi\left(\frac{dx}{ds}\delta x + \&c.\right)\right]_0^1 + \delta x_0 \int \frac{d\phi}{dx_0} ds + \&c. + \delta x_1 \int \frac{d\phi}{dx_1} ds + \&c.,$$

where the &c. indicate terms with y and z respectively written for x. The conditions at the ends are then found by equating this expression to zero.

594. The equations of the brachistochrone are found by equating the first variation of the time to zero. To determine whether this curve makes the time a maximum, a minimum, or neither, it is necessary to examine the terms of the second order. For this we refer the reader to treatises on the calculus of variations. In most cases there is obviously some one path for which the time is a minimum, and if our equations lead to but one path, that path must be a true brachistochrone. In other cases we can use Jacobi's rule. Let AB be the curve from A to B given by the calculus of variations. Let a second curve *of the same kind* but with varied constants be drawn through the initial point A and make an indefinitely small angle at A, with the curve AB. If they again intersect in some point C, the curve satisfies the conditions for a true minimum only if C be beyond B.

595. Theorem I. When the only force on the particle acts (like gravity) in a vertical direction, $\phi = 1/v$ is a function of z only, and the first two differential equations of the curve (Art. 591) admit of an immediate integration. Remembering that $dx/ds = \cos\alpha$, $dy/ds = \cos\beta$, it follows that *the brachistochrone for a vertical force is such a curve that at every point* $v = a\cos\alpha$, $v = b\cos\beta$, *where* α, β *are the angles the tangent makes with any two horizontal straight lines, and* a, b *are the two constants of integration.* By equating the two values of v and integrating, we see that the brachistochrone is a plane curve.

596. Theorem II. Let X, Y, Z be the components of the impressed forces, the mass being unity; then since $\frac{1}{2}v^2 = U + C$, we have $X = \frac{1}{2}dv^2/dx$, &c. The differential equations of the brachistochrone therefore become

$$\frac{d}{ds}\left(\frac{1}{v}\frac{dx}{ds}\right) + \frac{X}{v^3} = 0, \quad \frac{d}{ds}\left(\frac{1}{v}\frac{dy}{ds}\right) + \frac{Y}{v^3} = 0, \quad \&c. = 0 \ldots(1).$$

Let λ, μ, ν be the direction cosines of the binormal, then since the binormal is perpendicular both to the tangent and the radius of curvature

$$\lambda\frac{dx}{ds} + \mu\frac{dy}{ds} + \nu\frac{dz}{ds} = 0, \qquad \lambda\frac{d^2x}{ds^2} + \mu\frac{d^2y}{ds^2} + \nu\frac{d^2z}{ds^2} = 0 \ldots\ldots(2).$$

Using the values of X, Y, Z given in (1) we find

$$\lambda X + \mu Y + \nu Z = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

the resultant force is therefore perpendicular to the binormal, and its direction lies in the osculating plane.

Let $l = \rho \dfrac{d^2x}{ds^2}$, $m = \rho \dfrac{d^2y}{ds^2}$, &c. be the direction cosines of the positive direction of the radius of curvature, then

$$-\frac{lX + mY + nZ}{v^3} = \frac{1}{v}\frac{l^2 + m^2 + n^2}{\rho} + \frac{d}{ds}\left(\frac{1}{v}\right)\left\{l\frac{dx}{ds} + \&c.\right\}.$$

Since the radius of curvature is at right angles to the tangent, the last term is zero, and we have

$$lX + mY + nZ = -\frac{v^2}{\rho} \quad\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

This equation proves that *in any brachistochrone the component of the impressed forces along the radius of curvature is equal to minus the component of the effective forces in the same direction.*

597. *To find the pressure on the constraining curve.* Let F_1, F_2 be the components of the impressed forces in the directions of the radius of curvature and binormal. Let R_1, R_2 be the pressures on the particle in the same directions. Then by Art. 526

$$v^2/\rho = F_1 + R_1, \quad 0 = F_2 + R_2.$$

In a brachistochrone $F_2 = 0$ and $F_1 = -v^2/\rho$, hence $R_2 = 0$ and $R_1 = -2F_1$.

598. *To find a dynamical interpretation of Theorem II.*

We see by referring to the equations of motion in Art. 597, that if we changed the sign of F_1, the component of pressure R_1 would be zero, and the path would then be free. We also suppose the tangential component of force to remain unchanged so that the velocity is not altered. It follows immediately, that *a brachistochrone and a free path may be changed, either into the other, by making the resultant force at each point act at the same angle to the same direction of the tangent as before, but on the other side, and still in the osculating plane.* In this comparison the velocities of the particle, when free and when constrained, are equal at the same point of the path, i.e. $v' = v$.

599. Theorem III. The equations of motion of a particle P constrained to describe the brachistochrone are

$$\frac{d}{ds}\left(\frac{1}{v}\frac{dx}{ds}\right)=\frac{d}{dx}\left(\frac{1}{v}\right),\quad \frac{d}{ds}\left(\frac{1}{v}\frac{dy}{ds}\right)=\frac{d}{dy}\left(\frac{1}{v}\right),\ \&c.$$

If we now write $vv'=k^2$ or, which is the same thing $vds=k^2dt'$, where $v'=ds/dt'$, the first of these equations becomes

$$\frac{d}{ds}\left(v'\frac{dx}{ds}\right)=\frac{dv'}{dx}.$$

Now $v'dx/ds$ being the x component of the velocity, is equal to dx/dt'. Multiplying by v' or ds/dt', the equations take the form

$$\frac{d^2x}{dt'^2}=\frac{1}{2}\frac{dv'^2}{dx},\quad \frac{d^2y}{dt'^2}=\frac{1}{2}\frac{dv'^2}{dy},\ \&c.$$

These are the equations of motion of a free particle P' moving along the same path with a velocity v' and occupying the position x, y, z at the time t'. It follows that *the brachistochrone from point to point in a field $U+C$ is the same as the path of a free particle in a field $U'+C'$, provided* $U'+C'=\dfrac{k^4}{4}\dfrac{1}{U+C}$; i.e. $v'=\dfrac{k^2}{v}$.

To understand better the relation between the two fields of force we notice that if X, X' be the components of force in any the same direction at the same point,

$$X=\frac{dU}{dx},\quad X'=\frac{dU'}{dx},\quad \therefore\ X'=-X\left(\frac{k}{v}\right)^4.$$

We also notice that $dt'/dt=v/v'$.

600. This theorem is useful, as it enables us to apply to a brachistochrone the dynamical rules we have already studied for free motion. It also enables us to express at once the fundamental differential equations in polar or other co-ordinates.

The first theorem (Art. 595) follows at once from the third, for when the force is vertical we see by resolving horizontally that $v'\cos\alpha$ is constant. Since $v'=k^2/v$, this gives the result.

To deduce the second theorem, we notice that in the free motion $v'^2/\rho=F_1'$, where F_1' is the component of force along the radius of curvature. Using the theorems $v'=k^2/v$, $X'=-X(k/v)^4$, (where X is here F_1) this becomes $v^2/\rho=-F_1$.

601. *Ex.* 1. *To find the brachistochrone from one given curve to another, the acting force being gravity and the level of no velocity given.* The motion is supposed to be in a vertical plane.

Let the axis of x be at the level of no velocity and let y be measured downwards; then $v^2=2gy$. By Art. 595 the curve is such that $v=a\cos\alpha$. This gives $y=2b\cos^2\alpha$, where b is an undetermined constant. This is the well-known

equation of a cycloid, having its cusps at the level of no velocity. The radius of the generating circle and the position of the cusps on the axis are determined by the conditions that the cycloid cuts each of the bounding curves at right angles; Art. 592.

Ex. 2. If in the last example the bounding curves are two straight lines which intersect the axis of no velocity in the points L, L'; and make angles β, β' with the horizon, prove that the diameter $2b$ of the generating circle is $LL'/(\beta-\beta')$ and the distance of the cusp from L is $2b\beta$. Explain the results when the lines are parallel.

602. *Ex.* Show by using Jacobi's rule that the cycloid from one given point A to another B is a real minimum, the level of zero velocity being given (Art. 594).

The cycloid found by the calculus of variations passes through A and B and there is no cusp between these points. Describe a neighbouring cycloid passing through A and having its cusps on the same horizontal line, the radii of the generating circles being b and $b+db$. Since the base of a cycloid from cusp to cusp is $2\pi b$, it is easy to prove that the next intersection of the two curves lies in a vertical which passes between the two next cusps. The cycloids therefore cannot again intersect between A and B and the time from A to B must be a minimum. See also Art. 654.

603. *Ex.* Find the brachistochrone from one given curve to another when the acting force is gravity and the particle starts from rest at the upper curve.

Fixing the ends, it follows, from Art. 601, that *the brachistochrone is a cycloid having a cusp on the higher curve.* To determine the constants of the curve, we examine the part of δt due to the variation of the two ends. Let x_0, y_0; x_1, y_1 be the coordinates of the upper and lower ends, then $v^2=2g(y-y_0)$. By Art. 593 we have

$$\delta t=\phi\left\{\frac{dx}{ds}\delta x+\frac{dy}{ds}\delta y\right\}+\delta y_0\int\frac{d\phi}{dy_0}ds=0,$$

where $\phi=1/v$ and the expression is taken between limits. Now in our problem

$$-\frac{d\phi}{dy_0}=\frac{d\phi}{dy}=\frac{d}{ds}\left(\phi\frac{dy}{ds}\right),$$

by using the differential equation of the brachistochrone in Art. 591. We therefore have

$$\left[\phi\left(\frac{dx}{ds}\delta x+\frac{dy}{ds}\delta y\right)\right]_0^1-\delta y_0\left[\phi\frac{dy}{ds}\right]_0^1=0.$$

Remembering that $\phi=1/v$ and $v=a\cos\alpha$, this takes the form

$$[\delta x+\tan\alpha\,\delta y]_0^1-\delta y_0[\tan\alpha]_0^1=0.$$

When we fix the lower end, we have, since y is measured downwards, $\delta x_1=0$, $\delta y_1=0$. Hence

$$-(\delta x_0+\tan\alpha_0\,\delta y_0)-\delta y_0(\tan\alpha_1-\tan\alpha_0)=0 \quad\text{.................}\quad (1).$$

When we fix the upper end, $\delta x_0=0$, $\delta y_0=0$;

$$\therefore\ \delta x_1+\tan\alpha_1\,\delta y_1=0 \quad\text{................................}\quad (2).$$

The last of these two equations proves that *the brachistochrone cuts the lower curve at right angles,* while the first, giving $\delta y_0/\delta x_0=\delta y_1/\delta x_1$, proves that *the tangents to the bounding curves at the points where the brachistochrone meets them are parallel.*

604. *Ex.* 1. A particle falls from rest at a fixed point A to a fixed point C, passing through another point B; find the entire path when the time of motion is a minimum, (1) supposing B to be a fixed point, (2) supposing B constrained to lie on a given curve. [Math. Tripos, 1866.]

The paths from A to B, B to C are cycloids having their cusps on a level with the point A. It is supposed that there is no impact at B in passing from one cycloid to the next. The particle describes a small arc of a curve of great curvature and moves off along the next cycloid without loss of velocity.

We have yet to find the position of B when it is only known to lie on a given curve. Taking the origin at A, and the axis of z vertically downwards, we have $v^2=2gz$. The time is given by

$$\sqrt{(2g)}\,t=\int_A^B \frac{ds}{\sqrt{z}}+\int_B^C \frac{ds'}{\sqrt{z'}},$$

where accents refer to the lower cycloid.

$$\therefore \sqrt{(2g)}\,\delta t=\left[\frac{1}{\sqrt{z}}\left(\frac{dx}{ds}\,\delta x+\frac{dy}{ds}\,\delta y+\frac{dz}{ds}\,\delta z\right)\right]_A^B+\left[\frac{1}{\sqrt{z'}}\left(\frac{dx'}{ds'}\,\delta x+\frac{dy'}{ds'}\,\delta y+\frac{dz'}{ds'}\,\delta z\right)\right]_B^C=0,$$

by Art. 592. Let (α, β, γ), $(\alpha', \beta', \gamma')$, (θ, ϕ, ψ) be the direction angles of the tangents at B to the two cycloids and to the constraining curve. Then remembering that A and C are fixed points and that B is varied on the curve, we have

$$(\cos\alpha\cos\theta+\cos\beta\cos\phi+\cos\gamma\cos\psi)-(\cos\alpha'\cos\theta+\cos\beta'\cos\phi+\cos\gamma'\cos\psi)=0.$$

It follows that *the tangent to the locus of B makes equal angles with the tangents to the two cycloids AB, BC*. This determines the point B.

Ex. 2. Find the curve of quickest descent from a fixed point A to another C, supposing that a screen is interposed between A and C having a given finite aperture through which the path must pass. [So long as the curve AC can be arbitrarily varied the minimum curve is found by Arts. 591, 601. Hence if the single cycloid AC does not pass through the aperture, the minimum curve must pass through a point B on the boundary of the aperture. The curve then consists of two cycloids AB, BC, and the position of B is found by Ex. 1.] [Todhunter.]

605. *Ex.* 1. If the brachistochrone is a parabola when the force is parallel to the axis, prove that the magnitude of the force is inversely proportional to the square of the distance from the directrix. [This follows from the equation $v=a\cos\alpha$.] Prove also that the time of describing any arc PQ varies as the area contained by the focal radii, SP, SQ. [For $\cos\alpha$ varies as $1/p$, therefore dt varies as pds.] See also Art. 649.

Ex. 2. A point moves in a plane with a velocity always proportional to the curvature of the path, prove that the brachistochrone of continuous curvature between any two given points is a complete cycloid. [Math. Tripos, 1875.]

We here have $\int\rho ds=\int\phi dx$ a minimum, where $\phi=(1+y'^2)^2/y''$. The curve can be immediately found by using two rules in the calculus of variations. *First*, we have $\qquad \delta\int\phi dx=\phi\delta x+(Y_{\prime}-Y_{\prime\prime}{}')\,\omega+Y_{\prime\prime}\omega'+\int(-Y_{\prime}{}'+Y_{\prime\prime}{}'')\,\omega dx,$

where $Y_{\prime}$, $Y_{\prime\prime}$, are the partial differential coefficients of ϕ with regard to y', y''; $\omega=\delta y-y'\delta x$, and the part outside the integral sign is to be taken between limits. Also accents denote differentiation with regard to x. The extreme points being given, $\delta x=0$, $\delta y=0$ at each end. Hence exactly as in Art. 591, 592, the differential equation of the curve is $Y_{\prime}{}'-Y_{\prime\prime}{}''=0$ and $Y_{\prime\prime}=0$ at each end. This gives $Y_{\prime}-Y_{\prime\prime}{}'=A$.

Secondly, the calculus of variations gives also the integral

$$\phi = (Y_{,} - Y_{,,}')\, y' + Y_{,,} y'' + B.$$

Eliminating $Y_{,,}'$ between our two first integrals we find $\phi = Ay' + Y_{,,} y'' + B$, which contains two arbitrary constants A, B. Substituting for ϕ and $Y_{,,}$, this leads to $(1+y'^2)^2/y'' = \frac{1}{2}Ay' + \frac{1}{2}B$; $\therefore\ \rho\, ds = \frac{1}{2}A\, dy + \frac{1}{2}B\, dx$.

Taking the straight line $Ay + Bx = 0$ as an axis of ξ, this is equivalent to $\rho = C \sin\psi$ where $\sin\psi = d\eta/ds$ and C is a constant. This is the known equation of a cycloid. The condition $Y_{,,} = 0$ at each end gives y'' infinite and therefore $\rho = 0$. The cycloid is therefore complete.

Ex. 3. Prove that the differential equation of the brachistochrone from rest at one given point A to another point B, when *the length of the curve is also given*, is $\dfrac{a}{\sqrt{y}} + b = \sqrt{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}}$. [Airy's Tracts.]

To make $\int ds/v$ a minimum subject to the condition that $\int ds$ is a given quantity we use a rule supplied by the calculus of Variations. We make $\int(\lambda/v + 1)\, ds$ a minimum without regard to the given condition and finally determine the constant λ so that the arc has the given length.

606. Central force. *Ex.* 1. Prove that the brachistochrone for a central force F is given by $v = Ap$, where $\frac{1}{2}v^2 = \int F dr$ and p is the perpendicular from the centre of force on the tangent. The mass is unity, as is usual in these problems.

The brachistochrone is a free path for a particle moving about the same centre but with such a law of force that the velocity $v' = k^2/v$. Since $v'p = h$ by Art. 306, we have $v = Ap$.

When $F = \mu u^n$, and the velocity is equal to that from infinity, the differential equation $v = Ap$ can be integrated exactly as in Arts. 360, 363.

Ex. 2. Prove that the same path will be a brachistochrone for $F = \mu u^n$ and a free path for $F' = \mu' u^{n'}$ if $n + n' = 2$, provided the velocity in each case varies as some power of the distance.

For the brachistochrone and the free paths respectively, we have

$$v^2 = 2\mu u^{n-1}/(n-1), \quad v'^2 = 2\mu' u^{n'-1}/(n'-1).$$

These satisfy the condition $vv' = k^2$ if $n + n' = 2$, (Art. 599).

Ex. 3. Prove that the ellipse is a brachistochrone for a central force tending from the focus and equal to $\mu/(2a - r)^2$. [Townsend.]

The conic is a free path for a force μ/SP^2 tending to the focus S. Hence making the force act on the other side of the tangent as described in Art. 598, the conic is a brachistochrone for an equal force tending from the other focus H.

Ex. 4. Prove that the central repulsive force for the brachistochronism of a plane curve varies as $d\,(p^2)/dr$, the circle of zero velocity being given by the vanishing of p.

Prove that the cissoid $x(x^2 + y^2) = 2ay^2$ is brachistochronous for a central repulsive force from the point $(-a, 0)$ which at the distance r from that point is proportional to $r/(r^2 + 15a^2)^2$, the particle starting from rest at the cusp.

[Math. Tripos, 1896.]

Ex. 5. Prove that the lemniscate of Bernoulli can be described as a brachistochrone in a field of potential μr^6, r being measured from the node of the lemniscate, and find the necessary velocity. [See Arts. 320, 606, Ex. 2.]

[Math. Tripos, 1893.]

Ex. 6. A particle, acted on by a central attractive force whose accelerating effect at a distance r is $\dfrac{\mu r}{(a^2+r^2)^2}$, a being a constant, is projected from a given point with the velocity from infinity. Prove that the form of the groove in which it must move in order to arrive at another given point in the shortest possible time is a hyperbola whose centre coincides with the centre of force. [Math. Tripos.]

Ex. 7. Show that the force of attraction towards the directrix of a catenary, along perpendiculars to it, for which the catenary is a brachistochrone, will vary as the inverse cube of the perpendicular. [Coll. Ex. 1897.]

607. Brachistochrone on a surface. To find the brachistochrone on a given surface we require only a slight modification in the argument of Art. 591. Proceeding as before, we find

$$\delta t = \frac{1}{v}\left(\frac{dx}{ds}\delta x + \&c.\right) + \int (P\,\delta x + Q\,\delta y + R\,\delta z)\,ds,$$

where $P = \dfrac{d}{dx}\dfrac{1}{v} - \dfrac{d}{ds}\left(\dfrac{1}{v}\dfrac{dx}{ds}\right)$, with similar expressions for Q and R. Since δt is zero for all variations of the curve on the surface, we must have

$$P\,\delta x + Q\delta y + R\,\delta z = 0.$$

If $f(x, y, z) = 0$ is the equation of the surface, the variations are connected by the one equation

$$f_x\delta x + f_y\delta y + f_z\delta z = 0,$$

where suffixes imply partial differential coefficients. We must therefore have $P/f_x = Q/f_y = R/f_z$. The equations of a brachistochrone on the surface $f(x, y, z) = 0$ are therefore given by

$$\left(\frac{d}{dx}\frac{1}{v} - \frac{d}{ds}\frac{dx}{vds}\right)\Big/ f_x = \left(\frac{d}{dy}\frac{1}{v} - \frac{d}{ds}\frac{dy}{vds}\right)\Big/ f_y = \left(\frac{d}{dz}\frac{1}{v} - \frac{d}{ds}\frac{dz}{vds}\right)\Big/ f_z.$$

If the brachistochrone is to begin and end at given bounding curves drawn on the surface, we equate to zero the integrated part of δt, taken between the limits. Fixing the ends in turn, we see that at each end the cosine of the angle between the tangents to the curve and to the boundary is zero (Art. 592). *The brachistochrone therefore cuts the boundaries at right angles.*

608. By writing $v = k^2/v'$ as in Art. 599 these equations may be put into the form

$$\left(\frac{d^2x}{dt'^2} - \frac{dU'}{dx}\right)\Big/ f_x = \left(\frac{d^2y}{dt'^2} - \frac{dU'}{dy}\right)\Big/ f_y = \left(\frac{d^2z}{dt'^2} - \frac{dU'}{dz}\right)\Big/ f_z.$$

These are the equations of motion of a particle moving freely on the constraining surface. It follows that *the brachistochrone from point to point on a constraining surface in a field $U+C$ is a free path on the same surface in a field $U'+C'$, where*

$$\tfrac{1}{2}v^2 = U + C, \qquad \tfrac{1}{2}v'^2 = U' + C', \qquad vv' = k^2.$$

The relation between the component forces in any direction is $F' = -F\left(\frac{k}{v}\right)^4$.

Ex. If the particle is constrained by a smooth wire to describe the brachistochrone on the surface without a change in the field of force, prove that

$$-v^2 \sin\chi/\rho = G, \quad v^2\cos\chi/\rho = H + R, \quad -2G = R_2,$$

where H, G are the components of the impressed forces along the normal to the surface, and that tangent to the surface which is perpendicular to the path, and R, R_2 are the components of the pressure in the same directions. Also ρ is the radius of curvature of the path, and χ the angle the osculating plane makes with the normal to the surface.

The first is obtained by transforming the equation of motion of a free particle P', viz. $v'^2 \sin\chi/\rho = G'$ by the rule given above, the others then follow from the ordinary equations of motion of the particle P.

609. We may also sometimes find the brachistochrone on a given surface by making a comparison with the brachistochrone on some other more suitable surface.

Let us derive a second surface from the given one by writing for the coordinates x, y, z of any point P some functions of ξ, η, ζ, the coordinates of a corresponding point Q. Let these functions be such that

$$(dx)^2 + (dy)^2 + (dz)^2 = \mu^2\{(d\xi)^2 + (d\eta)^2 + (d\zeta)^2\},$$

where μ is a function of ξ, η, ζ. Geometrically this equation implies that every elementary arc ds drawn from a point P on the surface bears the same ratio to the corresponding arc $d\sigma$ drawn from Q, viz. the ratio $\mu : 1$.

The brachistochrone on the given surface is found by making t a minimum, where

$$t = \int \frac{ds}{v} = \int \frac{\mu\, d\sigma}{v},$$

and the velocity v of P is some given function of the coordinates of P.

Expressing v in terms of ξ, η, ζ, this integral implies that the corresponding curve on the derived surface is also a brachistochrone, the velocity v' being given by $v' = v/\mu$. The work functions for the motions of P and Q are respectively $v^2 = 2(U + C)$ and $U' = (U + C)/\mu^2$.

If we arrange matters so that μ/v is constant, the velocity on the second surface is constant. *The brachistochrones on the given surface then correspond to geodesics on the derived surface.*

This comparison assists us in determining the point on a brachistochrone with one end given at which the time ceases to be a minimum.

The derived surface may be obtained in many ways, for example by using the method of inversion. The theory of this surface is also used in making maps; see the United States Coast Survey, *Craig's treatise on Projections*. The application to brachistochrones is given by Darboux in his *Théorie générale des Surfaces*.

Ex. A particle P moves on a sphere under the action of a centre of repulsive force situated at a point O on the surface, and the velocity v at any point distant

r from O is $v = Ar^2$. Prove that the brachistochrone from one given point to another is a circle whose plane passes through O.

Inverting the sphere with regard to O, the diameter $2a$ being the constant of inversion, the derived surface is a tangent plane. The curve is traced out by Q, usually called *the stereographic projection* of that traced by P. The ratio of the elementary arcs described by P and Q are in the ratio $r^2 : 4a^2$. Hence if the path of P is a brachistochrone for a velocity $v = Ar^2$, that of Q is a brachistochrone for a uniform velocity. The path of Q is therefore a straight line and that of P is a circle. Another proof follows from Arts. 608, 318.

610. Bertrand's theorem. A series of brachistochrones is drawn on a given surface from a point A, and the arcs AB, AB', &c. are described in equal times, the velocity at A being given. Prove that the locus of B cuts all the brachistochrones at right angles.

The following amounts to Bertrand's proof. If possible let the angle $AB'B$ be acute. Drawing the arc BC so that the angle $CBB' > CB'B$, the sides of the triangle BCB' will *then* be elementary and the triangle may be regarded as rectilinear. It follows that the arc $CB' > CB$. The time of describing CB' is $>$ than that of describing CB because the velocity at every point in the neighbourhood of C is ultimately the same. The time of describing the line ACB is therefore less than that of describing AB' or AB. The path AB could not then be a brachistochrone. This proof is the same as that used by Salmon in his *Solid Geometry*, Art. 394, to prove the corresponding theorem for geodesics. Bertrand's theorem is now generally enunciated in a generalized form and to this we proceed in the next article.

611. A surface S_1 being given, let us draw from every point A on it that brachistochrone which starts off at right angles to the surface. Let lengths AB be taken along these lines so that the time t of transit from the surface along each is equal to a given quantity. The locus of the extremities B traces out a second surface which we may call S_2. By Art. 592, we have

$$\delta t = \delta\sigma_2 \cos\theta_2/v_2 - \delta\sigma_1 \cos\theta_1/v_1.$$

By construction $\cos\theta_1 = 0$ for each line and, since the times of describing neighbouring lines are equal, $\delta t = 0$. *It follows that the surface S_2 also cuts the lines at right angles.*

If the surface S_1 is an infinitely small sphere all the brachistochrones diverge from a given point A. The locus of the other extremities of the arcs drawn from A and described in equal times is therefore an orthogonal surface.

This proof may be applied to brachistochrones drawn on a given surface by expressing the conditions at the limits in Art. 607 in a form similar to that in Art. 592.

This theorem though enunciated for a brachistochrone applies generally to problems in the calculus of variations. The time t may stand for any integral of the form $\int\phi \,.\, ds$ where ϕ is a given function of x, y, z, and the curve is such that the integral is a minimum between any two points taken on it.

612. *Ex.* 1. Prove that the equations of a brachistochrone on a surface of revolution for a heavy particle with a given level of zero velocity are $r^2\dfrac{d\phi}{ds} = Av$,

$v^2=2gz$, where r, ϕ, z are cylindrical coordinates, z being measured downwards from the zero level. Prove also that the brachistochrone touches the meridian at the zero level.

Ex. 2. A heavy particle is projected from a given point along a smooth groove cut on the surface of a right circular cone whose axis is vertical and vertex upwards, with a velocity due to the depth from the vertex. Prove that, if it reach another given point not more than half-way round the cone in the least possible time, the curve of the groove must be such as would, if the cone were developed, become a parabola with the point corresponding to the vertex as focus.

[Math. Tripos, 1873.]

Ex. 3. Prove that the brachistochrone on a vertical cylinder for a heavy particle with a given level of zero velocity becomes the brachistochrone on a vertical plane when the cylinder is developed on the plane. [Roger.]

Ex. 4. Find the brachistochrone when the velocity at any point of space is proportional to the distance from a given straight line. Prove that the curve lies on a sphere and cuts all the circles whose planes are perpendicular to the given straight line at a constant angle, i.e., the curve is a loxodrome. [Tait.]

Motion of a particle relative to the earth.

613. Let O be any point on the surface of the earth and let λ be its latitude. Then λ is the angle which the normal to the surface of still water at O makes with the plane of the equator. Let $OL=b$ be a perpendicular from O on the axis of rotation. Let ω be the angular velocity of the earth, then the earth turns round its axis from west to east in the time $2\pi/\omega$.

As we intend to discuss the motion of a particle P relative to axes moving with the earth and having the origin at O, it is convenient to begin by reducing O to rest. We therefore apply to the particle P an accelerating force equal to $\omega^2 b$ and acting in the direction LO. We also apply an initial velocity equal to ωb opposite to the direction of motion of O, i.e. in a direction due westwards from O.

When the particle has been projected from the earth it is acted on by the attraction of the earth and the applied force $\omega^2 b$. The force usually called gravity is not the attraction of the earth, but is the resultant of that attraction and the centrifugal force. The form of the earth is such that at every point of its surface this resultant acts perpendicularly to the surface of still water. Let g be this force at the point O, then when the particle is at O, and O has been reduced to rest, the resultant force is represented by g.

When the moving point P has ascended to a height h, the attraction of the earth is altered and is nearly equal to $g(1-2h/a)$, where a is the radius of the earth. Since h is usually not more than a few hundred feet and a is roughly 4000 miles, it is obvious that the change in the value of gravity is so small that, *for a first approximation at least, we may regard gravity as a force constant in direction and magnitude.* Since $2\pi/\omega$ is 24 hours, we find that $\omega^2 a$ is nearly equal to $g/289$. Hence if we neglect gh/a *we must also neglect* $\omega^2 h$ *at all points near* O. The applied force $\omega^2 b$ is not neglected because at points near the equator b is nearly as large as the radius of the earth.

614. The equations of motion of a particle referred to axes moving with the earth have been already formed in Art. 499. We have here merely to express the components θ_1, θ_2, θ_3 in terms of the angular velocity ω of the earth. We then substitute the values of the space velocities u, v, w in the equations of the second order and neglect all terms of the form $\omega^2 x, \omega^2 y, \omega^2 z$. We thus find

$$u = \frac{dx}{dt} - y\theta_3 + z\theta_2, \qquad \frac{d^2x}{dt^2} - 2\frac{dy}{dt}\theta_3 + 2\frac{dz}{dt}\theta_2 = X,$$

$$v = \frac{dy}{dt} - z\theta_1 + x\theta_3, \qquad \frac{d^2y}{dt^2} - 2\frac{dz}{dt}\theta_1 + 2\frac{dx}{dt}\theta_3 = Y,$$

$$w = \frac{dz}{dt} - x\theta_2 + y\theta_1, \qquad \frac{d^2z}{dt^2} - 2\frac{dx}{dt}\theta_2 + 2\frac{dy}{dt}\theta_1 = -g + Z,$$

where X, Y, Z are the impressed forces other than gravity, the mass being unity.

615. It will clearly be convenient to choose as the axis of z the vertical at O. If the axis of x be directed along the meridian towards the south and the axis of y towards the west, we have

$$\theta_1 = \omega\cos\lambda, \quad \theta_2 = 0, \quad \theta_3 = -\omega\sin\lambda,$$

since λ is the latitude of the place.

It is sometimes necessary to take the axis of x inclined to the meridian at some angle β, the angle β being measured from the south towards the west. We then have

$$\theta_1 = \omega\cos\lambda\cos\beta, \quad \theta_2 = -\omega\cos\lambda\sin\beta, \quad \theta_3 = -\omega\sin\lambda.$$

616. If we wish the axes to move round the vertical with an angular velocity p, we have $\beta = pt + \epsilon$, where ϵ is some constant.

We then have

$$\theta_1 = \omega \cos \lambda \cos \beta, \quad \theta_2 = -\omega \cos \lambda \sin \beta, \quad \theta_3 = -\omega \sin \lambda + p.$$

The components θ_1, θ_2, θ_3 are not now constants, and in making the substitutions for u, v, w in the equations of motion their differential coefficients will not disappear. But if p be any small quantity of the same order as ω, these differential coefficients are of the order ω^2. The equations of motion will then be still represented by the forms given in Art. 614.

617. As in some few cases it is necessary to examine the terms which contain ω^2, we give the results of the substitution when the axis of z is vertical, while those of x, y point respectively southward and westward:

$$\frac{d^2x}{dt^2} + 2\omega \sin \lambda \frac{dy}{dt} - \omega^2 \sin^2 \lambda x - \omega^2 \sin \lambda \cos \lambda z = X,$$

$$\frac{d^2y}{dt^2} - 2\omega \cos \lambda \frac{dz}{dt} - 2\omega \sin \lambda \frac{dx}{dt} - \omega^2 y = Y,$$

$$\frac{d^2z}{dt^2} + 2\omega \cos \lambda \frac{dy}{dt} - \omega^2 \cos^2 \lambda z - \omega^2 \sin \lambda \cos \lambda x = -g + Z.$$

618. *Ex. A particle P is attached to a point A at the summit of a high tower and when in relative rest the particle is allowed to fall freely. The point A being at a height h vertically above O, it is required to find the point at which the particle strikes the horizontal plane at O.*

Taking the axes of x, y to point due south and west, the equations of motion are

$$x'' - 2y'\theta_3 = 0, \qquad y'' - 2z'\theta_1 + 2x'\theta_3 = 0, \qquad z'' + 2y'\theta_1 = -g,$$

where $\theta_1 = \omega \cos \lambda$, $\theta_3 = -\omega \sin \lambda$, and the accents denote d/dt (Art. 614). *We solve these by successive approximation.*

As a first approximation, we neglect the terms which contain ω. Remembering that initially x, y, x', y', z' are each zero and $z = h$, we arrive at $x = 0$, $y = 0$, $z = h - \frac{1}{2}gt^2$.

As a second approximation we substitute these values of x, y, z in the terms of the differential equations which contain θ or ω. We obtain after an easy integration

$$x = At + B, \qquad y = Ct + D - \tfrac{1}{3}gt^3\theta_1, \qquad z = Et + F - \tfrac{1}{2}gt^2.$$

The particle being initially in *relative rest* we have $x' = 0$, $y' = 0$, $z' = 0$, hence $A = 0$, $C = 0$, $E = 0$. The initial velocities *in space* are not required here, but (after O has been reduced to rest) these are given by $u = 0$, $v = -h\theta_1$, $w = 0$. To the value of v we may add the velocity of O, viz. $-\omega b$. Also when $t = 0$, we have $x = 0$, $y = 0$, $z = h$;

$$\therefore x = 0, \qquad y = -\tfrac{1}{3}gt^3\theta_1, \qquad z = h - \tfrac{1}{2}gt^2.$$

We see from the value of z that *the vertical motion is unaffected by the rotation of the earth.* The time of falling is given by $h = \frac{1}{2}gt^2$. Since $x = 0$ throughout the motion, the particle strikes the horizontal plane on the axis of y, and *there is*

no southerly deviation. Since $\theta_1=\omega\cos\lambda$ we have $y=-\frac{1}{3}g\omega\cos\lambda t^3$; *there is therefore a deviation towards the east* which is proportional to the cube of the time of descent. This deviation is greatest at the equator.

619. *Ex.* 1. Show that the path of a particle falling from relative rest is nearly the curve $325ay^2=\cos^2\lambda z^3$.

Ex. 2. A particle is projected vertically upwards in vacuo with a velocity V. Prove that when the particle reaches the ground there is no deviation to the south, and that the deviation to the west is $4\omega\cos\lambda V^3/3g^2$. [Laplace IV., p. 341.]

Ex. 3. A particle falls from relative rest at a point A situated at a height h above the point O. Supposing the resistance of the air to be represented by κv where v is the velocity and κ a small quantity, find the effect on the easterly deviation.

Measuring z upwards and neglecting the terms $x'\theta_3$, $y'\theta_1$, as we now know that they are of the order ω^2 (Art. 618), the equations of motion become

$$y''-2\theta_1 z'=-\kappa y', \qquad z''=-g-\kappa z'.$$

The vertical motion is sensibly the same as if the earth were at rest. Substituting $z'=-gt$ in the first equation,

$$y''+\kappa y'=-2g\theta_1 t, \qquad \therefore\ y'+\kappa y=-g\theta_1 t^2;$$

$$\therefore\ y=-\frac{1}{\delta+\kappa}g\theta_1 t^2=-\frac{1}{\delta}\left(1-\frac{\kappa}{\delta}+\frac{\kappa^2}{\delta^2}-\&c.\right)g\theta_1 t^2,$$

where $\delta=d/dt$. This leads at once to $y=-\frac{1}{3}g\theta_1 t^3\left(1-\dfrac{3\kappa}{4}t\right)$. *The easterly deviation is therefore slightly diminished by the resistance of the air.*

Ex. 4. Prove that, if the attraction of the earth on the falling particle were represented by $X=-gx/a$, $Y=-gy/a$, $Z=-g(1-2z/a)$, the time of falling from rest at a height h, as deduced from the equations of Art. 614, would be increased by the inappreciable fraction $5h/6a$ of itself. Thence show that the easterly deviation is not perceptibly altered.

Ex. 5. *The southern deviation.* A particle falls from relative rest at a point A situated on the vertical at a point O on the surface of the earth. Let the ~~southern~~ horizontal component of the attraction of the earth be represented by

$$-X=\sin\lambda\cos\lambda\,(Ax+Cz),$$

where A and C are very small functions of the ellipticity and the angular velocity of the earth, the point O having been reduced to rest. Prove that the southern deviation measured on the tangent plane at O is $\sin\lambda\cos\lambda gt^4\left(\frac{2}{3}\omega^2+\frac{5}{24}C\right)$.

This result is obtained by substituting the approximate values of y and z obtained in Art. 618 in the small terms given in Art. 617. Expressions for the components of the attraction of the earth are to be found in treatises on the "figure of the earth" (see Stokes' *Mathematical and Physical Papers*, vol. II. p. 142). These give approximately (after some reduction) $C=(2m-\epsilon)2g/a$, where $m=\omega^2a/g$ and $\epsilon=1/300$, hence $C=2\omega^2$ nearly.

620. Two cases of motion. Two special cases of the motion of a particle deserve attention; (1) when the particle in its motion does not deviate far from the vertical and (2) when the motion is nearly horizontal.

Supposing the axis of z to be vertical, the horizontal velocities dx/dt and dy/dt are small compared with the vertical velocity dz/dt in the first case. The products of the horizontal velocities by ω are therefore of a higher order of small quantities than the product of the vertical velocity by ω and should be neglected in a first approximation.

In the second case, on the contrary, dz/dt is small and we neglect its product by ω. The two sets of equations are therefore as follows (Art. 614):

$$\left.\begin{aligned}\frac{d^2x}{dt^2}+2\frac{dz}{dt}\theta_2&=X,\\ \frac{d^2y}{dt^2}-2\frac{dz}{dt}\theta_1&=Y,\\ \frac{d^2z}{dt^2}&=-g+Z.\end{aligned}\right\}\qquad \left.\begin{aligned}\frac{d^2x}{dt^2}-2\frac{dy}{dt}\theta_3&=X,\\ \frac{d^2y}{dt^2}+2\frac{dx}{dt}\theta_3&=Y,\\ \frac{d^2z}{dt^2}-2\frac{dx}{dt}\theta_2+2\frac{dy}{dt}\theta_1&=-g+Z.\end{aligned}\right\}$$

We notice that when the motion is nearly vertical the components θ_1, θ_2 enter into the equations, while θ_3 does not appear until we proceed to higher approximations. It is therefore the component of the angular velocity about a tangent to the earth which affects the motion.

On the other hand when the motion of the particle is nearly horizontal it is the component of the earth's rotation about the vertical, viz. θ_3, which plays the principal part.

If we compare the x and y equations for the case in which the motion is nearly horizontal with those given in Art. 614, when the square of ω is neglected we see that they express the motion of a particle moving freely in space but referred to axes which turn round the vertical with an angular velocity θ_3. If, as is generally the case, the forces X, Y are either zero or independent of the changes of the nearly constant quantity z, *we can thus obtain these equations in an elementary way.* The particle moves freely in space, unaffected by the rotation of the earth, but the axes of reference move round the vertical and leave the particle behind. This geometrical interpretation of the equations may be made more evident by considering some simple cases.

621. As an example *consider the case of a pendulum.* When the bob makes small oscillations the motion is nearly horizontal. To construct the motion we suppose the pendulum to oscillate freely in space (with the proper initial conditions). This oscillation is left behind by the earth, and the effect is that the plane of

oscillation appears to revolve about the vertical with an angular velocity equal and opposite to the vertical component of the earth's angular velocity. The plane of oscillation therefore turns from west to south with an angular velocity $\omega \sin \lambda$. This problem is more fully considered in Art. 624.

622. Flat trajectories. A bullet is projected from a gun, situated at the point O, with a great velocity V, in a direction making a small angle α with the horizon so that *the trajectory is nearly flat*. It is required to find the motion.

The initial velocity of the bullet in space (after O has been reduced to rest) is V. After leaving the gun the bullet describes a parabolic path in space, while the axes of reference turn with the earth round the vertical at O, and the bullet is left behind by the axes (Art. 620). Supposing that the initial plane of xz contains the direction of projection, the coordinates of the bullet at the time t are evidently

$$x = Vt \cos \alpha, \qquad y = -x\theta_3 t \text{ where } \theta_3 = -\omega \sin \lambda.$$

The deviation y is therefore always to the right of the plane of firing in the northern hemisphere, and to the left in the southern hemisphere. If R be the range the whole deviation is $Rt\omega \sin \lambda$. We notice also that the deviation y is independent of the azimuth of the plane of firing, and that the time of describing a given distance x is independent of the rotation of the earth.

The third equation of motion (Arts. 614, 615) gives

$$\frac{d^2z}{dt^2} = -g + 2\theta_2 \frac{dx}{dt}, \qquad \therefore\ z = Vt \sin \alpha - \tfrac{1}{2}gt^2 - V\omega t^2 \cos \alpha \cos \lambda \sin \beta,$$

where $\theta_2 = -\omega \cos \lambda \sin \beta$ and β is the angle the plane of firing makes with the meridian. The vertical deviation of the bullet from its parabolic path at the moment of reaching a target distant x from the gun is therefore $-xt\omega \cos \lambda \sin \beta$.

623. Deviation of a projectile. *Ex.* A particle is projected with a velocity V in a direction making an angle α with the horizontal plane, and the vertical plane through the direction of projection makes an angle β with the plane of the meridian, the angle β being measured from the south towards the west. If x is measured horizontally in the plane of projection, y horizontally in a direction making an angle $\beta + \frac{1}{2}\pi$ with the meridian, and z vertically upwards from the point of projection, prove that

$$x = V \cos \alpha t + (V \sin \alpha t^2 - \tfrac{1}{3}gt^3)\, \omega \cos \lambda \sin \beta,$$
$$y = (V \sin \alpha t^2 - \tfrac{1}{3}gt^3)\, \omega \cos \lambda \cos \beta + V \cos \alpha t^2 \omega \sin \lambda,$$
$$z = V \sin \alpha t - \tfrac{1}{2}gt^2 - V \cos \alpha t^2 \omega \cos \lambda \sin \beta,$$

where λ is the latitude of the place, and ω the angular velocity of the earth.

Prove also (1) that the increase of range on the horizontal plane through the point of projection is $4\omega \sin \beta \cos \lambda \sin \alpha\, (\frac{1}{3} \sin^2 \alpha - \cos^2 \alpha)\, V^3/g^2$,

(2) that the deviation to the right of the plane of projection is

$$4\omega \sin^2 \alpha\, (\tfrac{1}{3} \cos \lambda \cos \beta \sin \alpha + \sin \lambda \cos \alpha)\, V^3/g^2,$$

and (3) that the time T of flight is decreased by $2T \cos \alpha \cos \lambda \sin \beta\, V\omega/g$.

It is not usual in practical gunnery to take account of the rotation of the earth except when V is very great, and then only the terms containing V are perceptible.

624. Disturbance of a pendulum. A particle of mass m is suspended by a fine wire of length l from a point O fixed relatively to the earth, and being drawn aside, so that the wire

makes a small angle α with the vertical at O, is let go. It is required to find the motion; see Art. 621.

The equations of motion are those given in Art. 614. Taking the axis of z vertical and the origin at the position of equilibrium of the mass m we see that the ordinate z is less than $l(1-\cos\alpha)$, and the terms of the form $\theta\, dz/dt$ are of the order $l\omega\alpha^2$: these we shall reject. Let us also make the axes of x, y turn slowly round the vertical with such an angular velocity p relatively to the earth that $\theta_3 = -\omega\sin\lambda + p$ becomes zero, as explained in Art. 616. The equations of motion are now

$$\left.\begin{aligned} \frac{d^2x}{dt^2} = -\frac{T}{m}\frac{x}{l}, \quad \frac{d^2y}{dt^2} = -\frac{T}{m}\frac{y}{l}, \\ \frac{d^2z}{dt^2} - 2\frac{dx}{dt}\theta_2 + 2\frac{dy}{dt}\theta_1 = -g + \frac{T}{m}\frac{l-z}{l} \end{aligned}\right\} \quad \ldots\ldots\ldots(1),$$

where T is the tension of the string, and θ_1, θ_2 have the values given in Art. 616.

The third equation proves that the tension T differs from mg by quantities of the order $l\omega\alpha$ at least. Since x/l and y/l are of the order α, and we have agreed to reject terms of the order $\omega\alpha^2$, we must put $T = mg$ in the two first equations.

Since the two first equations are independent of ω, the motion of a real pendulum when affected by the rotation of the earth is the same as that of an ideal pendulum, unaffected by the rotation, but whose path, viewed by a spectator moving with the earth, appears to turn round the vertical with an angular velocity $p = \omega\sin\lambda$ in a direction south to west.

If $ln^2 = g$, the solutions of the equation are clearly

$$x = A\cos(nt + C), \quad y = B\sin(nt + D)\ldots\ldots\ldots(2).$$

It appears that the time of oscillation, viz. $2\pi/n$, is unaffected by the rotation of the earth. To determine the constants of integration, we notice that when the particle is drawn aside from the vertical and not yet liberated, it partakes of the velocity of the earth and has therefore a small velocity relative to the axes. This is equal to $-l\alpha\omega\sin\lambda$ and is transverse to the plane of displacement. Taking the plane of displacement as the plane of xz at the time $t = 0$, the initial conditions are

$$x = l\alpha, \quad y = 0, \quad dx/dt = 0, \quad dy/dt = -l\alpha\omega\sin\lambda.$$

It is then easy to see that

$$A = l\alpha, \quad Bn = -l\alpha\omega \sin\lambda, \quad C = 0, \quad D = 0.$$

The particle therefore describes an ellipse whose semi-axes are A and $-B$. Since the ratio of the axes, viz. $\omega \sin\lambda \sqrt{(l/g)}$ is very small, the ellipse is very elongated and the particle appears to oscillate in a vertical plane. The effect of the rotation of the earth is to make this plane appear to turn round the vertical with an angular velocity $\omega \sin\lambda$.

625. It is known that, independently of all considerations of the rotation of the earth, the path of the bob of a pendulum is approximately an ellipse whose axes have a small nearly uniform motion round the vertical. This progression of the apses vanishes when the angle subtended at the point of suspension by either axis of the ellipse is zero; see Art. 566. As the presence of this progression will complicate the experiment, it is important (1) that the angle of displacement should be small, (2) that the pendulum when drawn aside should be liberated without giving the bob more transverse velocity than is necessary. This is usually effected by fastening the bob when displaced to some point fixed in earth by a thread, and when the mass has come to apparent rest it is set free by burning the thread. The progression of the apses due to the angular magnitude of the displacement is in the opposite direction to that caused by the rotation of the earth.

The advantage of using a long pendulum is that the linear displacement of the bob may be considerable though the angular displacement of the wire is very small. The bob should also be of some weight, for otherwise its motion would be soon destroyed by the resistance of the air; Art. 113.

626. As we have rejected some small terms it is interesting to examine if these could rise into importance on proceeding to solve the equations (1) to a second approximation. To determine this we substitute the first approximation of Art. 624 (2) in the differential equations. The third equation shows that $T/m - g$ has two sets of terms. First, there are terms independent of ω which lead to the solution already obtained in Art. 555, and need not be again considered here. Next, there are terms which contain ω as a factor and have the form $\sin(nt \pm \beta)$ where $\beta = pt$, Art. 616. These when multiplied by x/l or y/l give no terms of the form $\sin nt$ or $\cos nt$. None of the terms which contain ω can rise into importance (Art. 303).

627. The idea of proving the rotation of the earth by making experiments on falling bodies originated with Newton. But more than a hundred years elapsed before any observations of value were made. In 1791 Guglielmini of Bologna made some experiments in a tower 300 feet high. The liberation of the balls was effected by burning the thread by which they were suspended, and this was not done until they had entirely ceased to vibrate as observed by a microscope. The vertical was determined by a plumb line, but he had to wait several months before it came to rest. The results were disappointing for they showed a deviation towards the south nearly as great as that towards the east. This discrepancy was due to two causes, (1) the numerous apertures in the walls of the tower caused slight winds, (2) the vertical was not ascertained until a change in the seasons had

altered its position. Other experiments were made by Benzenberg about 1802 in Hamburg, but Reich's experiments in 1831—3 in the mines of Freiberg are generally considered to be the most important. The height of the fall was $158\frac{1}{2}$ metres and the mean of 106 experiments gave a deviation to the east of $28\frac{1}{3}$ millimetres, the deviation to the south being about a twentieth of that towards the east. These were the experiments that Poisson selected to test the theory; he showed that the observed easterly deviation was within a thirtieth of that given by calculation. Poisson also investigates the general equations of motion of a particle relative to the earth and obtains equations equivalent to those given in Art. 617. He then applies them to a variety of problems. *Journal de l'école polytechnique*, 1838.

The defect of experiments on falling bodies is the smallness of the quantities to be measured. In 1851 Foucault invented a new method; he showed that the plane of oscillation of a simple pendulum appeared to rotate round the vertical with an angular velocity equal and opposite to the component of the earth's angular velocity. The advantage of this method is that the experiment can be continued through several hours, so that the slow deviation of the pendulum can be (as it were) integrated through a time long enough to make the whole displacement very large. Foucault's experiment was widely repeated with many improvements. Among English experiments we may mention those by Worms in 1859 at King's College, London, in Dublin by Galbraith and Haughton, at Bristol, at Aberdeen, at Waterford in 1895. The accuracy of the method is such that it is possible to deduce the time of rotation of the earth. Foucault's observations gave 23^h, 33^m, 57^s, while the repetition of the experiment at Waterford led to 24^h, 7^m, 30^s, the true time lying between the two (see *Engineering*, July 5, 1895). Though the experiment can be easily tried when only the general result is required, yet many difficulties arise when the deviation has to be found with accuracy. Indeed Foucault admitted that it was only after a long series of trials that he made the experiment succeed (see *Bulletin de la Société Astronomique de France*, Dec. 1896).

Inversion and Conjugate functions.

628. Inversion*. Let a point P of unit mass move under the action of forces whose potential in polar coordinates is $U = f(r, \theta, \phi)$. Produce any radius vector OP of the path to Q, where $OP \,.\, OQ = k^2$; the locus of Q is called the inverse path of that of P and any two points thus related are called inverse points. Let $OP = r$, $OQ = \rho$.

Let P', Q' be two other inverse points near the former, then since $OP \,.\, OQ = OP' \,.\, OQ'$, a circle can be described about the quadrilateral $PQP'Q'$. The elementary arcs PP', QQ' are therefore ultimately in the ratio $r : \rho$. If the points P, Q move so as

* The reader may consult a paper by Larmor in *The Proceedings of the London Mathematical Society*, vol. xv. 1884. The principle of least action is there applied to both the method of Inversion and that of Conjugate functions.

to be always inverse points, their velocities u, u_1, are connected by the equation $u/u_1 = r/\rho$.

The position of the point P in space is determined either by the quantities (ρ, θ, ϕ) or (r, θ, ϕ). Choosing the former as the coordinates, the Lagrangian equations of the motion of P are deduced from

$$T = \tfrac{1}{2}u^2 = \tfrac{1}{2}\frac{k^4}{\rho^4}(\rho'^2 + \rho^2\theta'^2 + \rho^2 \sin^2\theta\phi'^2),$$

$$U + C = f\left(\frac{k^2}{\rho}, \theta, \phi\right) + C.$$

These equations contain only the polar coordinates of Q. They primarily give the motion of a point Q describing the inverse path in such a manner that P and Q are always at inverse points.

Let us now transpose the factor k^4/ρ^4 from T to U. We then have (Art. 524)

$$T_2 = \tfrac{1}{2}(\rho'^2 + \&c.), \qquad U_2 = \frac{k^4}{\rho^4}\left\{f\left(\frac{k^2}{\rho}, \theta, \phi\right) + C\right\}.$$

The Lagrangian equations derived from these give the motion of a particle which describes the same path as that of Q, but in a different time. Let the particle be called Π. The form of T_2 shows that Π moves as a free particle, acted on by forces whose potential is U_2. We see also that the masses of the particles P and Π are equal. See also Art. 650, Ex. 2.

The path of either particle may be inferred from that of the other. *If the path of the particle P described with a work function $f(r, \theta, \phi) + C$ is known, then the other particle* Π, *if properly projected, will describe the inverse path, with a work function*

$$U_2 = \frac{k^4}{\rho^4}\left\{f\left(\frac{k^2}{\rho}, \theta, \phi\right) + C\right\}.$$

629. To find the relation between the velocities u, v of the particles P, Π, when passing through any inverse points P, Q, we notice that by the principle of vis viva $\tfrac{1}{2}u^2 = U + C$, $\tfrac{1}{2}v^2 = U_2$. It follows immediately that $v = uk^2/\rho^2$, and therefore that $ur = v\rho$. Since the planes of motion OPP', OQQ' coincide, *the angular momenta of the particles, when at inverse points of their paths, about every axis through the centre of inversion are equal.*

The constant C is determined by the consideration that the

known velocity u in the given path must satisfy the equation $\frac{1}{2}u^2 = U + C$.

The particles P, Π do not necessarily pass through inverse points of their respective paths at the same instant. Let t, τ be the times at which they pass through any pair P, Q, of inverse points; $t+dt$, $\tau+d\tau$ the times at which they pass through a neighbouring pair P', Q' of inverse points. Since the elementary arcs PP', QQ' are in the ratio $r:\rho$ while the velocities of P, Π are in the ratio $1/r : 1/\rho$, it follows by division that the elementary times dt, $d\tau$ are in the ratio $r^2 : \rho^2$. The relation between t and τ is found by integration from $\dfrac{dt}{d\tau} = \dfrac{r^2}{\rho^2}$. This agrees with the ratio given in Art. 524.

Supposing that the particles P, Π are projected from inverse points on their respective paths, their initial velocities must be inversely as their distances from the centre O of inversion. The initial directions of motion must be in the same plane and make supplementary angles with the radius vector which passes through both the initial positions.

630. If the particle P is constrained to move on a surface the argument needs but a slight alteration. The inverse point Q describes a curve which lies on the inverse surface. Let (ρ, θ, ϕ) be the polar coordinates of Q; then these may also be taken as the Lagrangian coordinates of P. Using the equation of the inverse surface, we have $\rho' = \dfrac{d\rho}{d\theta}\theta' + \dfrac{d\rho}{d\phi}\phi'$. Substituting the values of ρ, ρ' in the expressions for T and $U+C$ given in Art. 628, we proceed as before and arrive at similar results.

631. The Pressures. When the particles P, Π are constrained to move on a surface and the inverse surface respectively, *the pressures R_1, R_2, at any pair of inverse points are such that $R_1 r^3 = R_2 \rho^3$.*

To prove this we take any axis of z and resolve the forces on the particles perpendicularly to the meridian plane $zOPQ$, Art. 491. We then have

$$\frac{1}{r\sin\theta}\frac{dA}{dt} = \frac{1}{r\sin\theta}\frac{dU}{d\phi} + R_1\cos\alpha_1,$$
$$\frac{1}{\rho\sin\theta}\frac{dA}{d\tau} = \frac{1}{\rho\sin\theta}\frac{dU_2}{d\phi} + R_2\cos\alpha_2,$$

where A is the angular momentum of either particle about the axis of z, Art. 629, and dt, $d\tau$ are the times respectively occupied by the particles in passing from any pair of inverse points to an adjoining pair.

The forces R_1, R_2 act along the normals to the two surfaces. To understand the geometrical relations, we describe a sphere passing through P, Q and touching one surface. Then since the sphere has the property that for every chord the

product $OP \, . \, OQ$ is the same, the sphere will touch the inverse surface also. The normals therefore meet in the centre of the sphere and will make equal angles with every straight line perpendicular to the radius vector OPQ. The angles α_1, α_2 of resolution are therefore equal, if the reactions are taken positively towards the centre of the sphere.

Since $\rho^2 dt = r^2 d\tau$ and $\rho^2 U_2 = r^2 (U+C)$, we see at once that $r^3 R_1 = \rho^3 R_2$. Since $ur = v\rho$, we have $R_1/u^3 = R_2/v^3$, i.e. the pressures at inverse points are also as the cubes of the velocities.

Ex. Deduce from the relations $\rho^2 U_2 = r^2 (U+C)$, $\quad \frac{1}{2}u^2 = U + C$,

(1) that the parallel components G, G' of the impressed forces on the particles P, Π in any direction perpendicular to the radius vector are connected by the equation $\rho^3 G' = r^3 G$.

(2) that the radial components F, F', are connected by $\rho^3 F' + r^3 F = -4r^2 (U+C)$.

632. *Ex.* 1. The path of a free particle under the action of no forces is a straight line; in this case we have $u^2 = 2U = 2C$. By inversion the path of a free particle, when $v^2 = u^2 \dfrac{r^2}{\rho^2} = 2U_2$, is the inverse of a straight line, i.e. a circle passing through the origin. This gives $U_2 = Ck^4/\rho^4$, and the central force $F = 4Ck^4/\rho^5$. This is Newton's theorem that a circle can be described freely about a centre of force on the circumference whose attraction varies as the inverse fifth power of the distance.

Ex. 2. Show that a particle can describe the curve $\rho^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$ under the action of a force F in the origin which varies as $\dfrac{1}{\rho^5}\left\{\dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{3}{2}\dfrac{1}{\rho^2}\right\}$.

When the axes a, b of the curve are so unequal that their ratio is greater than $\sqrt{2}$, the force F changes from attraction to repulsion as the particle proceeds from the extremity of one axis to the other. Verify this by tracing the curve, and show that the curve is convex at the extremity of the lesser axis.

Ex. 3. Prove that the central forces F, F', under the action of which a curve and its inverse can be described about the centre of inversion are so related that $\dfrac{F'r'^3}{h'^2} + \dfrac{Fr^3}{h^2} = 2\dfrac{r^2}{p^2}$; show also that the velocities v, v' at inverse points are connected by $vr = v'r'$. [This follows easily from the expression for F given in Art. 310. When $h = h'$, Art. 629, this agrees with Art. 631, Ex.]

Ex. 4. A particle P moves on a sphere under the action of a centre of attractive force situated at a point O on the surface, and the velocity v at any point is B/r^2 where $r = OP$. Prove that the path is a circle whose plane passes through O.

Inverting the sphere, we find that the stereographic projection is a straight line. The result follows at once, see Art. 609.

633. Conjugate functions. Let the Cartesian coordinates (x, y), (ξ, η) of two corresponding points P, Q be so related that

$$x + yi = f(\xi + \eta i) \dots\dots\dots\dots\dots\dots\dots\dots (1),$$

where f is any real function and $i=\sqrt{(-1)}$. Expanding the right-hand side we have

$$x+yi=\phi(\xi,\eta)+\psi(\xi,\eta)\,i \quad\text{.................(2)},$$

where ϕ and ψ are real functions. The transformation is therefore effected by using the equations

$$x=\phi(\xi,\eta),\quad y=\psi(\xi,\eta)\quad\text{.................(3)},$$

the motion of P following geometrically from that of Q. Differentiating (1) we find

$$x'+y'i=f'(\xi+\eta i)\,.\,\{\xi'+\eta'i\},$$
$$x'-y'i=f'(\xi-\eta i)\,.\,\{\xi'-\eta'i\};$$
$$\therefore\ x'^2+y'^2=\mu^2\,.\,\{\xi'^2+\eta'^2\}\quad\text{.................(4)},$$

where μ^2 is a real positive quantity given by

$$\mu^2=f'(\xi+\eta i)\,.f'(\xi-\eta i)\quad\text{.................(5)}.$$

Let $U=F(x,y)$ be the work function of the forces which act on the particle P. The motions of P and Q may be deduced by the Lagrangian rule from

$$T=\tfrac{1}{2}\mu^2(\xi'^2+\eta'^2),\quad U=F\{\phi(\xi,\eta),\psi(\xi,\eta)\},$$

the constant of U being included in F for the sake of brevity.

Transposing the factor μ^2 to the work function, the equations

$$T_2=\tfrac{1}{2}(\xi_1^2+\eta_1^2),\quad U_2=\mu^2F(\phi,\psi),$$

give by the same rule the motion of a particle Π, whose mass is equal to that of P, which (when properly projected) will describe the same path as the point Q, but in a different time, Art. 524.

To find the relation between the velocities u, v of the particles P, Π at corresponding points of their paths, we observe that since $\frac{1}{2}u^2=U$, $\frac{1}{2}v^2=U_2$, the velocities are such that $v=\mu u$.

To find the ratio of the times dt, $d\tau$ we notice that, by (4), the corresponding arcs ds, $d\sigma$ are such at $ds=\mu d\sigma$, while $\mu u=v$. It follows by division that $dt=\mu^2 d\tau$.

634. *Ex.* It is known that a particle can describe the ellipse $x^2/a^2+y^2/b^2=1$, with a force tending to the centre equal to κr. It is required to find the conjugate path and law of force when we use the transformation $x\pm yi=(\xi\pm\eta i)^n/c^{n-1}$.

Let $x=r\cos\theta$, $y=r\sin\theta$; $\xi=\rho\cos\phi$, $\eta=\rho\sin\phi$; the equation of transformation then gives

$$r=\rho^n/c^{n-1},\qquad \theta=n\phi.$$

The equation of the path is therefore

$$\frac{\cos^2 n\phi}{a^2}+\frac{\sin^2 n\phi}{b^2}=\frac{c^{2n-2}}{\rho^{2n}}\,.$$

Also, $$\mu^2 = f'(\xi+\eta i) f'(\xi-\eta i) = n^2(\xi^2+\eta^2)^{n-1}/c^{2n-2};$$
$$\therefore\ \mu = n\rho^{n-1}/c^{n-1}.$$

Again in the elliptic orbit,
$$u^2 = 2(U+C) = \kappa(a^2+b^2-r^2).$$
Hence since $v=\mu u$,
$$v^2 = 2U_2 = \frac{n^2\kappa}{c^{2n-2}}\rho^{2n-2}\left(a^2+b^2-\frac{\rho^{2n}}{c^{2n-2}}\right);$$
$$\therefore\ -F = \frac{dU_2}{d\rho} = \frac{n^2\kappa}{c^{2n-2}}\rho^{2n-3}\left\{(n-1)(a^2+b^2) - \frac{(2n-1)\rho^{2n}}{c^{2n-2}}\right\}.$$
The ratio of the angular momenta, viz. $v\rho/ur$, is easily seen to be equal to n.

When $n=-1$, this transformation becomes $r=c^2/\rho$, $\theta=-\phi$. The transformation reduces to a simple inversion, except that ϕ is measured positively in the opposite direction to θ.

635. *Ex.* If the particle P is constrained to move on any given curve with a work function U, while the equal particle Π is constrained to move on the conjugate curve, with a work function $U_2=\mu^2 U$, the pressures R_1, R_2 on the two curves are in the ratio of the cubes of the velocities, i.e. $R_1/u^3 = R_2/v^3$. This gives also $R_2=\mu^3 R_1$.

The grouping of trajectories and Jacobi's solution.

636. The Cartesian equations of the motion of a free particle of unit mass are
$$x''=\frac{dU}{dx},\quad y''=\frac{dU}{dy},\quad z''=\frac{dU}{dz}\ \text{............(1)},$$
and to these we join the equation of energy
$$v^2 = x'^2+y'^2+z'^2 = 2U+2C\ \text{..............(2)}.$$
When the equations (1) have been integrated we have x, y, z expressed by three functions of t with six constants whose values become known when the initial values a, b, c of the coordinates and the initial velocities a', b', c' are given.

Since t enters into the equations (1) only in the form dt, the differential equations are not altered by writing $t+\epsilon$ for t. One of the constants of integration therefore enters into the solution as a mere addition to the time. When we eliminate the time we arrive at two equations which are the equations of all the possible trajectories in space. The constant ϵ disappears with t, and the equations of the possible trajectories contain five constants, of which the energy C may be regarded as one. To understand the

relations of these trajectories to each other it becomes necessary to group them into systems.

We first group the trajectories according to the values of the energy C. Taking any one group, having any given energy, the four remaining constants are determined for any special trajectory when the coordinates of some two points A, B arbitrarily chosen on it are given.

637. Action. If ds be an element of the arc of the trajectory, the integral $V=\int mvds$ is called *the action* as the particle passes from A to B. If mv^2 be the vis viva of the particle in any position we also have $V=\int mv^2dt$, the limits being the times t_1 and t of passing through A and B. When we are only concerned with the motion of a single particle, it is convenient to suppose its mass to be taken as unity.

Considering a single particle, let s be measured from A to B along the trajectory of least action and let the length AB be l. Let $A'B'$ be a neighbouring trajectory (Art. 590) from some point A' near A to a point B' near B. Proceeding as in Art. 591, writing v for ϕ, we find

$$\delta V=\left[v\frac{dx}{ds}\,\delta x+\&c.\right]+\int\left[\left\{\frac{dv}{dx}-\frac{d}{ds}\left(v\,\frac{dx}{ds}\right)\right\}\delta x+\&c.+\frac{dv}{dC}\,\delta C\right]ds\ldots(3),$$

where the part outside the integral is to be taken between the limits A and B and the energy C has been varied for the sake of generality. It is easy to deduce from the equations of motion (as in Art. 599) that the coefficients of δx, δy, δz inside the integral are zero. Also since $\frac{1}{2}v^2=U+C$, we have $vdv/dC=1$. Since $v\,dx/ds$ is the x component of the velocity we thus have

$$\delta V=x'\delta x+y'\delta y+z'\delta z-a'\delta a-b'\delta b-c'\delta c+(t-t_1)\,\delta C\ldots(4).$$

When we consider the motion of a system of particles, either constrained or free, and all taking different paths, it is more convenient to take t as the independent variable. Let us imagine the system to be moving in some manner which we will call the actual course. Let the work function of the field be U and let L be the Lagrangian function, then $L=T+U$ (Art. 506). Let θ_1, θ_2, &c. be any independent coordinates of the system, a_1, a_2, &c. their values in some position A occupied by the system at a time t_1. Then θ_1, θ_2, &c. are functions of t, whose forms it is our object to discover.

Let us next suppose the system to move in some varied manner, i.e. let the coordinates be functions of t slightly different from those in the actual course. By

the fundamental theorem* in the calculus of variations, we have

$$\delta\int L dt = \left[L\delta t + \Sigma \frac{dL}{d\theta'} \omega \right]_{t_1}^{t} + \int \Sigma \left(\frac{dL}{d\theta} - \frac{d}{dt}\frac{dL}{d\theta'} \right) \omega dt,$$

where $\omega = \delta\theta - \theta'\delta t$, Σ implies summation for all the coordinates θ_1, θ_2, &c. and the limits of integration are t_1 and t. Since each separate term inside the integral vanishes by Lagrange's equations (Art. 506), we have

$$\delta\int L dt = \left[(T+U)\delta t + \Sigma \frac{dT}{d\theta'} (\delta\theta - \theta'\delta t) \right]_{t_1}^{t}.$$

If the geometrical conditions do not contain the time explicitly T will be a homogeneous function of θ_1', θ_2', &c. (Art. 510) and therefore $\Sigma \frac{dT}{d\theta'}\theta' = 2T$. We also suppose that for each varied course the velocities are so arranged that the principle of energy holds, i.e. $T - U = C$, though C may be different for each course. Hence $L = 2T - C$, and $\delta\int C dt = \delta\{C(t-t_1)\}$. We now have the two equations

$$\delta\int L dt = -C(\delta t - \delta t_1) + \Sigma\left(\frac{dT}{d\theta'}\delta\theta\right) - \Sigma\left(\frac{dT}{da'}\delta a\right) \quad \text{..(A)}$$

$$\delta\int 2T dt = (t - t_1)\,\delta C + \Sigma\left(\frac{dT}{d\theta'}\delta\theta\right) - \Sigma\left(\frac{dT}{da'}\delta a\right) \quad \text{......(B).}$$

The action V of the system is the sum of the actions of the several particles. We therefore have $V = \int 2T dt$. When the system reduces to a single particle of unit mass $2T = x'^2 + y'^2 + z'^2$, and the equation (B) becomes the same as (4).

638. Let us consider the motion of a single free particle and *let the energy C be given*, therefore $\delta C = 0$. Let v_1, v_2 be the velocities at A, B; $\delta\sigma_1$, $\delta\sigma_2$ the displacements AA', BB'; θ_1, θ_2 the angles these displacements make with the positive directions of the tangents at A, B; then, as in Art. 592, (4) becomes

$$\delta V = v_2 \cos\theta_2 \delta\sigma_2 - v_1 \cos\theta_1 \delta\sigma_1 \quad \text{............(IV).}$$

* The proof of this theorem is as follows. We have

$$\delta\int L dt = \int(\delta L dt + L d\delta t) = [L\delta t] + \int(\delta L dt - dL\delta t).$$

Now L is a function of the letters typified by θ, θ',

$$\therefore\ \delta L = \Sigma(L_\theta \delta\theta + L_{\theta'}\delta\theta'), \qquad dL = \Sigma(L_\theta d\theta + L_{\theta'}\theta'' dt),$$

where suffixes imply partial differential coefficients. Since

$$\delta\frac{d\theta}{dt} = \frac{d\theta + d\delta\theta}{dt + d\delta t} - \frac{d\theta}{dt} = \frac{d\delta\theta}{dt} - \frac{d\theta}{dt}\frac{d\delta t}{dt},$$

$$\therefore\ \delta\theta' - \theta''\delta t = \frac{d}{dt}(\delta\theta - \theta'\delta t) = \omega',$$

substituting we find

$$\delta\int L dt = [L\delta t] + \int\Sigma(L_\theta \omega + L_{\theta'}\omega')\,dt.$$

Integrating the last term by parts we immediately obtain the theorem in the text.

Introducing the mass m, this may be read, *the change of the action in passing from one trajectory AB to a neighbouring one is the difference of the virtual moments of the momenta at the two ends.*

Taking any arbitrary surface which we may call S_1, let us group together all the trajectories which cut S_1 orthogonally, then $\cos\theta_1 = 0$. On each of these trajectories let us take the point B so that the action from the surface S_1 to B is some given quantity. As we pass from one trajectory to a neighbouring one, B traces out a second surface which we may call S_2, and at every point of S_2 we have $\delta V = 0$. It follows that for this surface (supposing it to be of finite extent) $\cos\theta_2$ is also zero. The trajectories therefore intersect the surface S_2 at right angles.

Considering all possible trajectories we first group them according to the value of the energy. We classify them again by selecting all those at right angles to some given surface. We have now a congruence of trajectories. The theorem just proved asserts that all these trajectories can be cut orthogonally by a system of surfaces. These orthogonal surfaces are such that, when any two are given, the action from one to the other is the same for all the trajectories. See Thomson and Tait, *Treatise on Natural Philosophy*, 1879, vol. I. Art. 332.

All possible trajectories may be grouped together in the manner just described in many different ways. One method is to select a surface intersecting all the trajectories. Each point of this surface may be regarded as the centre of an infinitely small sphere which all the trajectories intersect at right angles. The surface S_1 is then reduced to a collection of points occupying an arbitrary surface. This is the method of grouping adopted in Arts. 159, 330, 339, &c. By a different grouping we obtain different orthogonal surfaces.

639. These considerations lead us to a rule which is a special case of that given by Jacobi for the solution of dynamical problems. When this method is applied to the dynamics of a particle the orthogonal surfaces are investigated first and the trajectories are afterwards deduced. In the general case of a system of rigid bodies the interpretation is not so simple.

640. *Let the action V be expressed as a function of the energy C and of the coordinates (x, y, z), (a, b, c) of the particle in the two arbitrary positions B and A.* Then by the principles of the differential calculus,

$$dV = \frac{\delta V}{dx}\delta x + \frac{dV}{dy}\delta y + \frac{dV}{dz}\delta z + \frac{dV}{da}\delta a + \frac{dV}{db}\delta b + \frac{dV}{dc}\delta c + \frac{dV}{dC}\delta C \ldots (5),$$

the energy being varied for the sake of generality. Comparing this with the expression (4) (Art. 637) we see that

$$x' = \frac{dV}{dx}, \text{ \&c., \&c.,} \quad a' = -\frac{dV}{da}, \text{ \&c., \&c.,} \quad t - t_1 = \frac{dV}{dC} \ldots (6).$$

Substituting in the equation (2) of energy, we find

$$\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 = 2U + 2C, \quad \left(\frac{dV}{da}\right)^2 + \left(\frac{dV}{db}\right)^2 + \left(\frac{dV}{dc}\right)^2 = 2U_0 + 2C .. (7),$$

where U_0 is the value of U when we write for x, y, z their initial values a, b, c. These are called *the Hamiltonian equations of motion.*

It is obvious that if we can deduce from the equations (7) the proper form for the function V, the first set of (6) will give the component velocities of the particle and the second set will give the relations between the coordinates x, y, z and their initial values. The last equation will give the time.

Jacobi proved that it is not necessary to obtain the general integral of either differential equation. It is sufficient to discover one solution of the form

$$V = f(x, y, z, C, \alpha, \beta) + \gamma \ldots\ldots\ldots\ldots\ldots (8),$$

containing three new constants α, β, γ. He also proved that the introduction of the initial coordinates a, b, c into the expression for V is unnecessary. Instead of these he uses the two constants of integration here called α, β.

641. In the first differential equation (7) and in the complete integral (8), the quantities x, y, z are the independent variables. Jacobi's rule asserts that *if we establish the following relations between x, y, z and a new variable t, the equations of motion* (1) *will be satisfied.* These assumed relations are

$$\frac{df}{d\alpha} = -\alpha_1, \quad \frac{df}{d\beta} = -\beta_1, \quad \frac{df}{dC} = t + \epsilon \ldots\ldots\ldots (9),$$

where α_1, β_1, and ϵ are three new constants. These new relations make x, y, z functions of t, C and the five constants α, β, α_1, β_1, and ϵ.

To prove these relations we differentiate (9) with regard to t and thus arrive at three equations of the form

$$x'\frac{d^2f}{dx\,d\alpha}+y'\frac{d^2f}{dy\,d\alpha}+z'\frac{d^2f}{dz\,d\alpha}=0\ldots\ldots\ldots\ldots\ldots(10).$$

The other equations have β and C written for α, but in the third the zero on the right-hand side is replaced by unity. These equations determine x', y', z'.

Also since (8) is a solution of the first of the differential equations (7), it must satisfy that equation identically. We may therefore differentiate (7) *after substitution* with regard to each of the constants α, β, C. We thus arrive at three equations of the form

$$\frac{df}{dx}\frac{d^2f}{dx\,d\alpha}+\frac{df}{dy}\frac{d^2f}{dy\,d\alpha}+\frac{df}{dz}\frac{d^2f}{dz\,d\alpha}=0\ldots\ldots\ldots\ldots(11).$$

The other equations have β and C written for α, but in the third the zero is replaced by unity.

Comparing the three equations (10) with the three (11), we see at once that

$$x'=\frac{df}{dx},\quad y'=\frac{df}{dy},\quad z'=\frac{df}{dz}\ \ldots\ldots\ldots\ldots\ldots(12).$$

It also follows that

$$x''=\frac{d^2f}{dx^2}x'+\frac{d^2f}{dx\,dy}y'+\frac{d^2f}{dx\,dz}z'\ldots\ldots\ldots\ldots\ldots(13),$$

with similar expressions for y'', z''.

We may also differentiate (7) after substitution from (8) partially with respect to any one of the three variables x, y, z;

$$\therefore\ \frac{df}{dx}\frac{d^2f}{dx^2}+\frac{df}{dy}\frac{d^2f}{dx\,dy}+\frac{df}{dz}\frac{d^2f}{dx\,dz}=\frac{dU}{dx}.$$

Substituting from (12), the left-hand side becomes by (13) equal to x''. We therefore have

$$x''=\frac{dU}{dx},\quad y''=\frac{dU}{dy},\quad z''=\frac{dU}{dz},$$

which are the equations of motion (1).

642. Consider the system of surfaces defined by

$$f(x, y, z, C, \alpha, \beta) = K \text{.....................} (14),$$

where C, α, β are constants and K the parameter. The equations (12) prove that the direction of motion at any point is normal to that surface of the system which passes through the point. Thus *the surfaces* (14) *cut the trajectories at right angles.* These trajectories (with their parameters α_1, β_1) may be deduced from (14) by the rules given in the theory of differential equations or more easily by Jacobi's equations (9).

The trajectories in Jacobi's method are thus grouped together according to their orthogonal surfaces. By taking different complete integrals for (8), we group the same trajectories in different ways. Art. 638.

643. As an example which requires no long algebraical process, let us discuss the trajectories when the forces are absent. The Hamiltonian equation is

$$\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 = 2C \text{ . ... } (15).$$

One complete integral, suggested by the rules for solving differential equations, is

$$V = \{\alpha x + \beta y + \surd(1 - \alpha^2 - \beta^2) z\} \surd(2C) + \gamma \text{ . ... } (16),$$

another complete integral is

$$V = \{(x - \alpha)^2 + (y - \beta)^2 + z^2\}^{\frac{1}{2}} \surd(2C) \text{ .... } (17).$$

If we choose the first integral the surfaces $V = K$ are planes and the trajectories are grouped into systems of parallel lines, the lines taking all directions. If we choose the second integral, the surfaces $V = K$ are spheres having their centres on the plane of xy. The trajectories are grouped into systems of straight lines diverging from points on that plane.

To illustrate the use of equations (9) let us substitute in them the second integral. We have at once

$$\frac{x - \alpha}{r} = -\alpha_1, \quad \frac{y - \beta}{r} = -\beta_1, \quad \frac{r}{\surd(2C)} = t + \epsilon \text{ . . . . } (18),$$

where $r^2 = (x - \alpha)^2 + (y - \beta)^2 + z^2$. These evidently give a system of straight lines diverging from the point $x = \alpha$, $y = \beta$, $z = 0$, described with a velocity $\surd(2C)$.

644. When the coordinates chosen are not Cartesian the expression for the kinetic energy does not take the simple form given in (2). Let the kinetic energy T be given by

$$2T = P\theta'^2 + Q\phi'^2 + R\psi'^2 \text{...................} (19),$$

where P, Q, R are functions of the coordinates θ, ϕ, ψ. Let us now take as the Hamiltonian equation

$$\frac{1}{P}\left(\frac{dV}{d\theta}\right)^2 + \frac{1}{Q}\left(\frac{dV}{d\phi}\right)^2 + \frac{1}{R}\left(\frac{dV}{d\psi}\right)^2 = 2U + 2C \text{......} (20).$$

Proceeding exactly in the same way as before, we prove that if

$$V = f(\theta, \phi, \psi, C, \alpha, \beta) + \gamma \;\ldots\ldots\ldots\ldots\ldots (21),$$

be an integral of (20), the first integrals of the Lagrangian equations of motion (Art. 506), are

$$P\theta' = \frac{df}{d\theta}, \quad Q\phi' = \frac{df}{d\phi}, \quad R\psi' = \frac{df}{d\psi} \;\ldots\ldots\ldots (22).$$

The trajectories, &c. are given by

$$\frac{df}{d\alpha} = -\alpha_1, \quad \frac{df}{d\beta} = -\beta_1, \quad \frac{df}{dC} = t + \epsilon \;\ldots\ldots\ldots\ldots (23),$$

where α_1, β_1, and ϵ are new constants.

This enunciation includes the most useful cases of Jacobi's rule. But his method applies also to any dynamical system, in which T is a quadratic function of the velocities. For these generalizations we refer the reader to treatises on Rigid Dynamics.

645. *Ex.* 1. Apply Jacobi's rule to find the path of a projectile.

The Hamiltonian equation is

$$\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 = -2gy + 2C.$$

Separating the variables, we find that one complete integral is

$$V = \sqrt{(2\alpha)}\, x - \frac{1}{3g}(2C - 2\alpha - 2gy)^{\frac{3}{2}} + \gamma.$$

Ex. 2. Apply Jacobi's method to find the path of a particle in three dimensions about a fixed centre of force which attracts according to the Newtonian law.

Taking polar coordinates we have

$$2T = r'^2 + r^2\theta'^2 + r^2 \sin^2\theta\phi'^2, \qquad U = \frac{\mu}{r}.$$

The Hamiltonian equation (Art. 644) may be put into the form

$$\left\{r^2\left(\frac{dV}{dr}\right)^2 - 2\mu r - 2Cr^2\right\} + \left(\frac{dV}{d\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{dV}{d\phi}\right)^2 = 0.$$

If we equate these three expressions respectively to α, $-\alpha + \beta \operatorname{cosec}^2\theta$ and $-\beta \operatorname{cosec}^2\theta$, we obtain three differential equations in which the variables are separated and whose solutions satisfy the Hamiltonian equation. Let the integrals of these be $V = f_1(r, \alpha)$, $V = f_2(\theta, \alpha, \beta)$, $V = f_3(\phi, \beta)$. It is obvious that $V = f_1 + f_2 + f_3 + \gamma$ is a complete integral from which all the trajectories may be deduced.

Ex. 3. Apply Jacobi's method to find the motion of a particle in elliptic coordinates (λ, μ, ν) when the work function is

$$U = \frac{(\mu^2 - \nu^2) f_1(\lambda) + (\nu^2 - \lambda^2) f_2(\mu) + (\lambda^2 - \mu^2) f_3(\nu)}{(\lambda^2 - \mu^2)(\mu^2 - \nu^2)(\nu^2 - \lambda^2)}.$$

Taking the expression for T given in Art. 577, the Hamiltonian equation (Art. 644) after a slight reduction becomes

$$(\lambda^2 - h^2)(\lambda^2 - k^2)(\mu^2 - \nu^2)\left(\frac{dV}{d\lambda}\right)^2 + (\mu^2 - h^2)(\mu^2 - k^2)(\nu^2 - \lambda^2)\left(\frac{dV}{d\mu}\right)^2 + (\nu^2 - h^2)(\nu^2 - k^2)(\lambda^2 - \mu^2)\left(\frac{dV}{d\nu}\right)^2$$

$$= -2\{(\mu^2 - \nu^2) f_1(\lambda) + (\nu^2 - \lambda^2) f_2(\mu) + (\lambda^2 - \mu^2) f_3(\nu)\} - 2CD,$$

where $D = (\lambda^2 - \mu^2)(\mu^2 - \nu^2)(\nu^2 - \lambda^2)$. Since

$$(\mu^2 - \nu^2) + (\nu^2 - \lambda^2) + (\lambda^2 - \mu^2) = 0,$$

$$\lambda^2(\mu^2 - \nu^2) + \mu^2(\nu^2 - \lambda^2) + \nu^2(\lambda^2 - \mu^2) = 0,$$

$$\lambda^4(\mu^2 - \nu^2) + \mu^4(\nu^2 - \lambda^2) + \nu^4(\lambda^2 - \mu^2) = -D,$$

the differential equation is satisfied by assuming

$$(\lambda^2 - h^2)(\lambda^2 - k^2)\left(\frac{dV}{d\lambda}\right)^2 = -2f_1(\lambda) + \alpha + \beta\lambda^2 + 2C\lambda^4,$$

with similar expressions for $dV/d\mu$ and $dV/d\nu$. In these trial solutions the variables λ, μ, ν have been separated, the first containing λ, the second μ, and the third ν. Supposing the integrals to be $V = F_1(\lambda, \alpha, \beta, C)$, $V = F_2(\mu, \&c.)$, $V = F_3(\nu, \&c.)$, the required complete integral is then $V = F_1 + F_2 + F_3 + \gamma$. The solution then follows by simple differentiations with regard to the constants α, β, C.

This expression for U is given by Liouville in his *Journal*, vol. XII. 1847. He uses it in conjunction with Jacobi's solution.

We may also write the expression in a different form. Let p_1, p_2, p_3 be the perpendiculars from the origin on the tangent planes to the three confocals which intersect in any point, and let λ, μ, ν be as before the semi-major axes. We find by using the expressions for these perpendiculars in elliptic coordinates (Art. 577)

$$U = p_1^2 F_1(\lambda) + p_2^2 F_2(\mu) + p_3^2 F_3(\nu).$$

Taking $U = p^2 F(\lambda)$, (omitting the suffixes) we see at once that the level surfaces intersect the ellipsoids in the polhodes. The direction of the force at any point P is therefore normal to the polhode which passes through P. It may be shown by differentiation that the components, T and N, of the force, tangential and normal to the ellipsoid which passes through P, are

$$T = -2p^4 F(\lambda)\{S_8 - p^2 S_6^2\}^{\frac{1}{2}} = -2p^6 F(\lambda)\left\{\Sigma \frac{x^2 y^2 (\lambda^2 - b^2)^2}{\lambda^8 b^8}\right\}^{\frac{1}{2}},$$

$$N = 2p^5 F(\lambda) S_6 + \frac{p^3}{\lambda} F'(\lambda),$$

where $S_n = \frac{x^2}{\lambda^n} + \frac{y^2}{b^n} + \frac{z^2}{c^n}$. The Cartesian components X, Y, Z are

$$X = \frac{2p^4 x}{\lambda^2}\left\{-\frac{1}{\lambda^2} + 2p^2 S_6\right\} F(\lambda) + \frac{p^4 x}{\lambda^2}\frac{F'(\lambda)}{\lambda},$$

with similar expressions for Y and Z.

We may obtain simpler expressions by combining the three terms of U. Putting $f_1(\lambda) = -\lambda^{2n+4}$, $f_2(\mu) = -\mu^{2n+4}$, $f_3(\nu) = -\nu^{2n+4}$, we see that U is equal to the sum of the different homogeneous products of λ^2, μ^2, ν^2 of n dimensions, each product being taken with a coefficient unity. This symmetrical function of the roots of the cubic in Art. 576 may be expressed as a rational function of the coefficients. We thus find possible forms for U in Cartesian coordinates. For example, putting $f_1(\lambda) = -\lambda^6$ &c., we find

$$U = \lambda^2 + \mu^2 + \nu^2 = (x^2 + y^2 + z^2) + A.$$

As another example, put $f_1(\lambda) = -\lambda^8$ &c., we then have

$$U = \lambda^4 + \mu^4 + \nu^4 + \lambda^2\mu^2 + \mu^2\nu^2 + \nu^2\lambda^2$$
$$= (x^2 + y^2 + z^2)^2 + (x^2 + y^2 + z^2)(h^2 + k^2) + h^2y^2 + k^2z^2 + B,$$

where A and B are two constants.

646. Principle of least action. Let the extremities A, B of the trajectories be given and let the particle be constrained to move from one point to the other along a smooth wire, the energy being given, Art. 636. Of all the different methods of conducting the particle from A to B there may be one which is the trajectory the particle would take if unconstrained. We see by Art. 637 that for this course the value of δV is given by equation (4). But since the points A, B are fixed, δx, δy, δz vanish at each end. We therefore have $\delta V = 0$. It follows therefore that the free trajectory is such that the change of action in passing from it to any neighbouring constrained course is zero. *The action for a free trajectory with given energy is either a maximum, a minimum, or is stationary.*

Conversely, if the path from A to B is required which makes the action a max-min, the principles of the Calculus of Variations require that the coefficients of δx, δy, δz inside the integral (3) in Art. 637 should be zero, provided the geometrical conditions of the problem permit δx, δy, δz to have arbitrary signs. Assuming this, the vanishing of the coefficients leads, as already explained, to the equations of motion. The result is that *the free trajectory from A to B is then the path of max-min action given by the calculus of variations.*

A similar theorem holds for the motion of a system either free or connected by geometrical relations. Let any two configurations or positions A, B be given. If we conduct the system from A to B by any varied paths as described in Art. 637 we have (since the variations of the coordinates of these positions are zero)

$$\delta\int L\,dt = -C(\delta t - \delta t_1) \ldots\ldots (A), \qquad \delta\int 2T\,dt = (t - t_1)\,\delta C \ldots\ldots (B).$$

Let us now suppose that in these varied paths the particles, without violating the geometrical relations, are conducted with such velocities that *the energy $C = T - U$ has a given value,* (the same as in the actual course,) then $\delta C = 0$, and the equation (B) shows that *the action $\int 2T\,dt$ is a max-min or is stationary in the actual path.*

The equation (A) gives a companion theorem. Let us suppose that in the varied paths the particles are so conducted that *the time $t - t_1$ is equal to a given quantity, then $\int L\,dt$ is a max-min or is stationary.*

647. The action from one given point to another cannot be a real maximum if the velocity is always the same function of the position of the particle. Every element of either of the integrals $\int v^2 dt$ or $\int v ds$ is positive and therefore, whatever path from A to B may be taken, we can increase the whole action by conducting the particle along a sufficiently circuitous but neighbouring path. Thus, if C be any point on the free course AB we can conduct the particle along that course to C, then compel it to make a circuit, and after returning to the neighbourhood of C conduct it along the remainder CB of the free path. Additional positive terms are thus given to the integral and the action is increased. The energy of the motion is unaltered, but the time of transit is longer.

Since every element of the integral is positive, there must be some path joining A and B which makes the action a true minimum. If the theory of max-min in the Calculus of Variations gives only one path, that path must be a minimum.

648. It may be that there are several free paths by which the particle could travel from A to B. Selecting one of these, say ADB, we may ask if the action along it is a true minimum. Let a neighbouring free path starting from A (the energy being the same) intersect ADB in C. To simplify matters let no other free path intersect ADB nearer to A than C. If B lie between A and C there is only one free path from A to B which is in accordance with the principles of mechanics, and that path makes the action a true minimum; Art. 647. If B is beyond C, there are two neighbouring free paths from A to C. It may be proved that the action from A to B is not in general a true minimum, the action for some neighbouring courses being greater and for others less than for the free path AB (Art. 653).

649. It may be that there is no free path from A to B, yet there must be a path of minimum action. For example, a heavy particle projected from A with a given velocity can by a free path arrive only at such points as lie within a certain paraboloid whose focus is at A, Art. 159. The path of minimum action from A to a point B beyond the paraboloidal boundary is not a free path. When deduced from the Calculus of Variations it falls under the case mentioned in Art. 646. Its position is such that it cannot be varied arbitrarily on all sides, i.e. the signs of the variations δx, δy, δz are not arbitrary along the whole length of the course.

Such limitations exist when the path runs along the boundary of the field of motion (Art. 299). We therefore draw verticals from A and B to intersect the level of zero velocity (which in this case is the directrix) in C and D. Let us conduct the particle from A along AC to a point as near C as we please, and thence along a course coinciding indefinitely nearly with the directrix to a point as near D as we please. The particle is finally conducted along the vertical DB to the given point B. Throughout this course the velocity is always supposed to be $\sqrt{(2gz)}$ where z is the depth below the directrix. The velocity being ultimately zero along the directrix the whole action from A to B is reduced to the sum of the actions along the vertical paths AC, DB. The path close to the directrix cannot be varied arbitrarily, because the particle cannot be conducted above that level without making the velocity imaginary. This minimum path is therefore not given by the ordinary rules of the Calculus of Variations.

A similar anomaly occurs in the case of brachistochrones. The parabola is a brachistochrone when the force acts parallel to the axis and is such that the velocity is inversely proportional to the square root of the distance from the

directrix; Art. 605. The directrix being given in position, the initial and final points A, B of the course may be so far apart that no such parabola can be drawn. In this case the brachistochrone is found by conducting the particle along the vertical straight line AC in accordance with the given law of velocity, thence with an infinite velocity along the directrix CD, and finally along the vertical line DB to B.

The further discussion of these points is a part of the Calculus of Variations. Some remarks on the dynamics of the problem may be found in the author's *Rigid Dynamics*, vol. II. chap. x.

650. *Ex.* 1. Prove that the same path is a brachistochrone for $v^2=f(x, y, z)$ and a path of least action for $v'^2=A/f(x, y, z)$; Art. 599.

The brachistochrone is deduced from the calculus of variations by making $\int ds/v$ a minimum; the path of least action by making $\int v'ds$ a minimum. These must give the same curve if $v'=k^2/v$; (Jellett and Tait).

Ex. 2. Prove that, if a path be described by a particle P with such a work function that $v^2=f(r, \theta, \phi)$, the inverse path can be described by a particle Π with a velocity v', such that $v'^2=\dfrac{k^4}{\rho^4}f\left(\dfrac{k^2}{\rho}, \theta, \phi\right)$, where $r\rho=k^2$; Art. 628.

To find the first path we make $\int v ds$ a minimum. Since $ds'/ds=\rho/r$, the second path is found by making $\int v'ds\rho/r$ a minimum. These are the same integrals. This mode of proof applies equally whether the particle is free or constrained to move on a surface.

651. *Ex.* 1. Prove that in an elliptic orbit described about the focus S, the time is measured by the area described about the focus S and the action by the time described about the empty focus H.

If p, p' be the perpendiculars on the tangent from S and H, we know that $pp'=b^2$. Since $v=h/p$, the action $\int v ds$ becomes $\int p'ds \,.\, h/b^2$; the area described about H being $\frac{1}{2}\int p'ds$, the result follows at once. [Tait, *Dynamics of a particle.*]

Ex. 2. In an ellipse described about the centre C, perpendiculars PM, PN are drawn from P on the major and minor axes CA, CB, and A, B represent the elliptic areas PMA, $PNCA$ respectively. Prove that the action from A to P is

$$(a^2A+b^2B)\sqrt{\mu}/ab.$$

Ex. 3. Prove that the action in describing an arc of a central orbit is $\displaystyle\int\frac{h}{p}\left(1-\frac{p^2}{r^2}\right)^{-\frac{1}{2}}dr$. When the central force is $F=\mu/r^n$ and the initial velocity is that from infinity, prove also that the action is $\dfrac{2h}{n-3}\tan\dfrac{n-3}{2}\theta$, where θ is measured from the maximum or minimum radius vector; Art. 360.

Ex. 4. A heavy particle describes a parabola. Prove that the action from any point A to another B is κ times the sectorial area ASB, where S is the focus, $\kappa^2=16g/l$ and l is the semi-latus rectum.

Prove also that, if the chord AB pass through the focus, the action along the parabolic path is greater than that along the course AC, CD, DB where AC, BD are perpendiculars on the directrix. Arts. 159, 649.

652. *Ex.* 1. When a heavy particle is projected from a point A with a given velocity to pass through a point B, there are in general two possible parabolic paths. Prove that the action is a minimum along that parabola in which the arc AB is less than the arc AC where C is the other extremity of the chord drawn from A through the focus.

The action is a minimum when B is not beyond the intersection with the neighbouring parabola drawn from A; Art. 648. Since the chord of intersection ultimately passes through the focus of either of these neighbouring parabolas, Art. 159, the result given follows at once.

Ex. 2. When the force is central and varies according to the Newtonian law, there are in general two elliptic paths which a particle could take when projected from A with a given velocity to pass through B. Prove that the action is a minimum along that ellipse in which the arc AB is less than AC, where C is the other extremity of the chord drawn from A through the empty focus: Art. 339.

653. *Ex.* *A particle describes a circular orbit about a centre of force represented by $F=\mu/r^n$, situated in the centre O. It is required to find the change in the action when the particle is conducted with the same energy from a given point A to another B on the circle by some neighbouring path lying in the plane of the circle.*

Let a be the radius, then taking the normal resolution, the velocity $v_0=\sqrt{(\mu/a^{n-1})}$. The principle of energy for the varied path gives

$$\frac{v^2}{2}=\frac{\mu}{n-1}\frac{1}{r^{n-1}}+C.$$

Also $C=\frac{1}{2}\frac{n-3}{n-1}\frac{\mu}{a^{n-1}}$, since the energy C is the same for both paths.

Let the equation of the varied path be $r=a(1+\rho)$ where ρ is some function of θ. Substituting we find

$$v=v_0\{1-\rho+\tfrac{1}{2}(n-1)\rho^2+\dots\}\ \dots\dots\dots\dots \quad (1).$$

Here ρ is equivalent to the δr of the Calculus of Variations.

Since $(ds)^2=r^2(d\theta)^2+(dr)^2$, we find by the same substitution

$$\frac{ds}{d\theta}=a\left\{1+\rho+\frac{1}{2}\left(\frac{d\rho}{d\theta}\right)^2+\dots\right\}\ \dots\dots\dots\dots \quad (2).$$

The action therefore when θ increases from 0 to θ is

$$\int v\,ds=av_0\left\{\theta+\frac{1}{2}\int\left\{\left(\frac{d\rho}{d\theta}\right)^2-p^2\rho^2\right\}d\theta+\dots\right\}\ \dots\dots\dots \quad (3),$$

where $p^2=3-n$ as in Art. 367, and the limits are $\theta=0$ to θ. *By substituting for ρ the value corresponding to any assumed variation of the path, the change in the action follows immediately.*

If the particle starting from A were to describe a neighbouring free path with the same energy, we know by Art. 367 that the first intersection of the new path with the circle is at a point given by $\theta=\pi/p$ nearly.

We may easily deduce from the expression (3) that *the action from A to B is a*

true minimum if the angle $AOB<\pi/p$; see Art. 594, 648. To prove this we use an artifice due to Lagrange*. Since

$$\frac{d}{d\theta}(\lambda\rho^2)=2\lambda\rho\frac{d\rho}{d\theta}+\rho^2\frac{d\lambda}{d\theta} \quad\ldots\ldots\ldots\ldots\ldots(4),$$

where λ is an arbitrary function of θ, we may write the integral on the right-hand side of (3) in the form

$$I=-[\lambda\rho^2]+\int\left\{\left(\frac{d\rho}{d\theta}\right)^2+2\lambda\rho\frac{d\rho}{d\theta}+\left(\frac{d\lambda}{d\theta}-p^2\right)\rho^2\right\}d\theta.$$

The term $\lambda\rho^2$ taken between the limits is zero, since both paths begin at A and end at B. Let us choose the function λ so that

$$\lambda^2=\frac{d\lambda}{d\theta}-p^2, \quad \therefore \lambda=p\tan p(\theta-\alpha)\ \ldots\ldots\ldots\ldots\ldots(5),$$

then

$$I=\int\left(\frac{d\rho}{d\theta}+\lambda\rho\right)^2 d\theta\ \ldots\ldots\ldots\ldots\ldots(6).$$

Since this integral is essentially positive it follows from (3) that the action along every varied path from A to B is greater than that along the circle.

This argument requires that λ should not be infinite within the limits of integration. By taking $p\alpha=\frac{1}{2}\pi-\epsilon$ where ϵ is a quantity as small as we please the values of λ given by (5) can be made finite from $\theta=0$ to $\theta=\pi/p-\epsilon'$ where ϵ' is a quantity as small as we please. *The argument therefore requires that the point B should not make the angle $AOB>\pi/p$.*

When the angle AOB is greater than π/p we can prove that the action along some varied curves extending from A to B is less, and along others is greater, than that in the circle.

To prove this let us conduct the particle from A to B along the varied path whose equation is $\rho=L\sin g\theta$. Let β be the angle AOB, then since ρ vanishes at each end, g is arbitrary except that $g\beta$ is a multiple of π. Since $p\beta>\pi$ one value at least of g is less than p and the others are greater than p. Substituting in (3), we find that the integral is

$$I=\int\left\{\left(\frac{d\rho}{d\theta}\right)^2-p^2\rho^2\right\}d\theta=\frac{L^2\beta}{2}(g^2-p^2)\ \ldots\ldots\ldots\ldots\ldots(7),$$

the limits being $\theta=0$ to $\theta=\beta$. The smaller values of g make I negative, while the greater values (which correspond to the more circuitous routes) make I positive. The conclusion is that *when the angle $AOB>\pi/p$, the action along the circle is not a true minimum.*

654. *Ex. A particle moves in a plane with a velocity $v=\phi(x, y)$ beginning at a given point A and ending at B. The path taken being that of minimum action, it is required to find in Cartesian coordinates the equation of the path and the change of action when the path is varied in an arbitrary manner.*

Let the elementary action $v\,ds=\phi\sqrt{(1+y'^2)}\,dx$ be represented by $f(x, y, p)\,dx$, where p has been written for $y'=dy/dx$. Then writing $y+\delta y$, $p+\delta p$ for y and p,

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* Lagrange *Théorie des fonctions Analytiques* 1797. He refers to Legendre, *Memoirs of the Academy of Sciences* 1786, and adds that it must be shown that λ does not become infinite between the limits of integration. Not being able to settle this question, he just missed Jacobi's discovery. See also Todhunter's *History of the Calculus of Variations*, page 4.

(but not varying x) the whole increase of action on the varied curve is by Taylor's theorem,

$$\delta A = \int [f_y \delta y + f_p \delta p + \tfrac{1}{2} \{f_{yy} (\delta y)^2 + 2 f_{yp} \delta y \delta p + f_{pp} (\delta p)^2\} + \&c.]\, dx,$$

where suffixes as usual represent partial differential coefficients. Integrating the second term by parts, as in Art. 591, we have

$$\delta A = [f_p \delta y] + \int \{(f_y - f_p') \delta y + \&c.\}\, dx,$$

where the part outside the integral, being taken between fixed limits, is zero, and accents denote total differentiation with regard to x. The path of minimum action is found by equating the coefficient of δy to zero, Art. 591. This path is therefore given by

$$f_y - f_p' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

and the change of action in any varied path by

$$\delta A = \tfrac{1}{2} \int [f_{yy} (\delta y)^2 + 2 f_{yp} \delta y \delta p + f_{pp} (\delta p)^2]\, dx \ldots\ldots\ldots\ldots\ldots \quad (2).$$

To find the path in Cartesian coordinates we integrate the equation (1). This can only be effected when the form of the function ϕ is given. The integration presents only those difficulties which are discussed in treatises on differential equations. We now proceed to find the change in the action given by (2).

To determine the sign of δA, we write (2) in the form

$$\delta A = [\lambda (\delta y)^2] + \tfrac{1}{2} \int [(f_{yy} - 2\lambda') (\delta y)^2 + 2 (f_{yp} - 2\lambda) \delta y \delta p + f_{pp} (\delta p)^2]\, dx \ldots \quad (3),$$

where the term outside the integral is zero, provided λ does not become infinite between the limits of integration.

Let $y = F(x, c_1, c_2)$ be the integral of (1), then changing the constants into $c_1 + \alpha$, $c_2 + \beta$ where α, β are indefinitely small,

$$y + \delta y = F + \frac{dF}{dc_1} \alpha + \frac{dF}{dc_2} \beta \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4),$$

is also a solution of (1). We choose the constants c_1, c_2 so that the curve $y = F$ passes through the limiting points A and B. Making the varied curve (4) also pass through A, we have an equation to find β/α. Hence

$$\delta y = \alpha \left(\frac{dF}{dc_1} + \frac{dF}{dc_2} \frac{\beta}{\alpha} \right) = u \quad \ldots\ldots\ldots\ldots\ldots \quad (5),$$

is the equation of a neighbouring path of minimum action beginning at A and making a small arbitrary angle with the path AB, the magnitude of the angle depending on that of α. If C is the *first point of intersection* of these two paths, then u is not zero between A and C.

Differentiating (1) we see that $\delta y = u$ satisfies the equation

$$f_{yy} \delta y + f_{yp} \delta p - \frac{d}{dx} (f_{yp} \delta y + f_{pp} \delta p) = 0;$$

$$\therefore \left(f_{yy} - \frac{d}{dx} f_{yp} \right) u = \frac{d}{dx} (f_{pp} u') \quad \ldots\ldots\ldots\ldots\ldots \quad (6).$$

Returning to the integral (3) let us choose λ so that

$$(f_{yp} - 2\lambda) u = -f_{pp} u' \quad \ldots\ldots\ldots\ldots\ldots \quad (7).$$

Substituting in (6) we find

$$\left(f_{yy} - \frac{d}{dx} f_{yp} \right) u = -\frac{d}{dx} (f_{yp} - 2\lambda) u$$

$$= -\left(\frac{d}{dx} f_{yp} - 2 \frac{d\lambda}{dx} \right) u + \frac{(f_{yp} - 2\lambda)^2 u}{f_{pp}},$$

the last term being obtained by substituting for u' from (7). This becomes

$$\left(f_{yy} - 2\frac{d\lambda}{dx}\right) f_{pp} = (f_{yp} - 2\lambda)^2 \quad\text{.............................(8).}$$

The quantity under the integral sign in (3) is therefore a perfect square. Remembering (7) we see that

$$\delta A = \frac{1}{2}\int f_{pp}\left\{\delta p - \frac{u'}{u}\delta y\right\}^2 dx \quad\text{.............................(9).}$$

The value of λ is by (7)

$$2\lambda = \left(\frac{dv}{dy}p + \frac{v}{1+p^2}\frac{u'}{u}\right)\frac{1}{\sqrt{(1+p^2)}} \quad\text{.......................(10).}$$

Hence in order that both λ and the subject of integration in (9) may be finite *it is necessary that u should not vanish between the limits of integration.* The second limiting point B must therefore not be beyond C. It is supposed that v and dv/dy are finite between the same limits. See Art. 648.

Supposing this condition to be satisfied, every term of the integral (9) is positive if f_{pp} is positive from A to B. Since $f_{pp} = v\,(1+p^2)^{-\frac{3}{2}}$, and the velocity v is supposed to keep one sign throughout the motion, this condition also is satisfied. *The change of action caused by a variation of path is therefore always positive and its amount is determined by* (2) or (9).

This investigation can be applied to brachistochrones and may also be extended to any cases in which the subject of integration, viz. $f(x, y, p)$, is a function only of the coordinates y, x, and the first differential coefficient. In order that the course AB given by (1) should be a true minimum, no variation must exist which can make δA negative. The conditions for this are (1) the point B must not be beyond C, as explained in Arts. 594, 648, (2) the differential coefficient d^2f/dp^2 must be positive throughout the whole course AB.

If d^2f/dp^2 were negative for any portion PQ of the course given by (1), let us vary the remaining portions AP, QB so that δy is as nearly equal to u as we please, the portion PQ being varied in some other manner. In this variation such prominence is given to the negative elements of the integral (9) that δA is made negative. It is also evident from (7) that λ is finite if d^2f/dp^2, $d^2f/dp\,dy$ are finite.

A SWARM OF PARTICLES.

Note on Art. 414.

THE argument will be made more complete if we suppose that the boundary of the swarm is an ellipsoid instead of a sphere. Owing to the manner in which the forces of attraction depend on the shape of the swarm, the results for an ellipsoid are not altogether the same as those for a sphere.

Taking the same axes as before, the coordinates of the projection of any particle P on the plane of motion of the centre are $r+\xi$, η, while ζ is the distance of P from that plane. Treating the ellipsoid as homogeneous and of density D, the component attractions of the swarm at any internal point are $A\xi$, $B\eta$, $C\zeta$, where A, B, C are functions of the ratios of the axes of the bounding ellipsoid and their sum is $4\pi D$.

The equations (1) of Art. 414 are slightly modified by having their last terms replaced by $-A\xi$, $-B\eta$; and instead of (3) we have

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2}-2n\frac{d\eta}{dt}+(A-3n^2)\,\xi&=0\\ \frac{d^2\eta}{dt^2}+2n\frac{d\xi}{dt}+B\eta\qquad\quad&=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I)}.$$

The equation for ζ is evidently

$$\frac{d^2\zeta}{dt^2}=-\frac{M\zeta}{r^3}-C\zeta=-(n^2+C)\,\zeta\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(II)}.$$

Putting $\xi=a\cos(pt+\alpha)$, $\eta=b\sin(pt+\alpha)$, and $\zeta=c\sin(qt+\gamma)$ we find by proceeding as in Art. 414,

$$\{p^2-(A-3n^2)\}\{p^2-B\}-4p^2n^2=0,\quad q^2=n^2+C\ldots\ldots\ldots\ldots\text{(III)}.$$

The condition for stability is therefore $A>3n^2$.

In an ellipsoid $A>B$ if the axis in the direction of ξ is less than that in the direction of η. It follows that if the axis of ξ is the least axis, A is greater for an ellipsoid than for a sphere. The swarm is therefore more stable for an ellipsoidal than for a spherical swarm provided the least axis of the ellipsoid is placed along the radius vector from the sun.

Let us suppose that all the particles are describing the same principal oscillation. The projections of their paths on the plane $\xi\eta$ are therefore given by $\xi=a\cos\theta$, $\eta=b\sin\theta$, where $\theta=pt+\alpha$. These paths are coaxial ellipses described in the same periodic time $2\pi/p$, the semi-axes of any ellipse being a, b. By substituting these values of ξ, η in the second of equations (I), we find $\dfrac{a}{b}=\dfrac{p^2-B}{-2np}$; it follows that all the ellipses are similar to each other. There will therefore be no collisions between the particles.

The ratio of the axes of the ellipses is not altogether arbitrary. By using (III) we find

$$\left(\frac{a}{b}\right)^2 = \frac{p^2 - B}{p^2 - (A - 3n^2)}, \quad \therefore p^2(a^2 - b^2) - Aa^2 + Bb^2 = -3a^2n^2,$$

where A, B and therefore p^2 are known functions of the ratios of the axes of the ellipsoid. We may deduce from the values of A, B given in the theory of Attractions that Aa^2 is less or greater than Bb^2 according as a^2 is greater or less than b^2. It then follows from this equation that in both the principal oscillations the axis of the ellipsoid in the direction of the radius vector from the sun is less than the axis of the ellipsoid in the direction of motion of the centre.

If P, Q, R be any three particles describing similar co-axial ellipses in the same time with an acceleration tending to their common centre, it is not difficult to prove that the area of the triangle PQR is constant throughout the motion. Let us apply this theorem to the motion of the projections of the particles on the plane of $\xi\eta$. Joining adjacent triads of particles, we divide the whole area into elementary triangles. If the swarm is homogeneous, the areas of these triangles are initially equal and we see that they will remain equal throughout the motion. The swarm will therefore remain homogeneous.

Consider next the motions of the particles perpendicular to the plane of $\xi\eta$. These are harmonic oscillations and are all described in the same time $2\pi/q$. The amplitude of each oscillation is the ordinate of the ellipsoid corresponding to the ellipse described by the projection and this is constant for the same particle. The distance between two adjacent particles moving in the same ordinate in the same direction is increasing or decreasing according as they are approaching or receding from the plane of $\xi\eta$. As there are as many particles approaching as receding, the uniformity of the density is not affected by this motion.

When both the principal oscillations are being described simultaneously the state of the motion becomes more complicated. The outer boundary is not strictly ellipsoidal, being dependent on both the states of motion. Since also the rotations in the principal oscillations are in opposite directions, we can no longer neglect the collisions between the particles.

To take account of the collisions we must have recourse to a statistical theory analogous to the kinetic theory of gases. But this would lead us too far from the methods of this treatise.

For an example of the application of the kinetic theory the reader is referred to a memoir by G. H. Darwin, *On the mechanical conditions of a swarm of meteorites, &c.*, *Phil. Trans.* 1889. He supposes a number of meteorites to be falling together from a condition of wide dispersion and to have not yet coalesced into a system of a sun and planets. No account is taken of the rotation of the system.

Callandreau has discussed the case in which a comet, regarded as a spherical swarm of particles, is heterogeneous, the density being a function of the distance from the centre. The effect of a passage near Jupiter has also been taken into account. See his *Étude sur la théorie des comètes périodiques*. He considers it probable that the periodic comets are undergoing a gradual disintegration and he points out that according to this hypothesis a few comets captured by the action of Jupiter could by repeated subdivisions produce all those known to exist. See *The Observatory*, Feb. 1898.

LAGRANGE'S EQUATIONS.

Note on Art. 524.

This rule may be put into another form. We know that if $L=T+U+C$ be the Lagrangian function and θ, ϕ, &c. the coordinates, the equations of motion are

$$\frac{d}{dt}\frac{dL}{d\theta'}=\frac{dL}{d\theta}, \qquad \frac{d}{dt}\frac{dL}{d\phi'}=\frac{dL}{d\phi}, \text{ \&c.} \quad\ldots\ldots\ldots\ldots\ldots (1).$$

We now see that we may use the same equations, if we substitute

$$L=\frac{T_2}{M}+M(U+C)\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where M is any arbitrary function of the coordinates θ, ϕ, &c. which we may find suitable when solving the equations.

The expression for T_2 differs from T only in the fact that the differential coefficients are taken with regard to a different independent variable, which has been represented by τ. Thus

$$T=\tfrac{1}{2}A_{11}\left(\frac{d\theta}{dt}\right)^2+A_{12}\frac{d\theta}{dt}\frac{d\phi}{dt}+\text{\&c.}; \qquad T_2=\tfrac{1}{2}A_{11}\left(\frac{d\theta}{d\tau}\right)^2+A_{12}\frac{d\theta}{d\tau}\frac{d\phi}{d\tau}+\text{\&c.}\ldots(3).$$

When the equations have been solved the paths of the particles are found by eliminating τ without enquiry into its meaning.

The equation of energy is supposed to be $T-U=C$; the constant C is therefore known when the initial values of θ, ϕ, &c., θ', ϕ', &c. are given.

We notice that one solution must be analogous to that given by the principle of vis viva. We therefore have $\frac{T_2}{M}=M(U+C)$. Since this must agree with the equation $T=U+C$, it immediately follows that $T=T_2\left(\frac{d\tau}{dt}\right)^2$, $T_2=M^2T$. The relation between τ and t is therefore $Md\tau=dt$.

When the paths of the particles are alone required, we may eliminate the time from the Lagrangian equations by using a new function instead of the Lagrangian function.

In this method we choose some one coordinate θ to be the independent variable and regard the others ϕ, ψ, &c. as unknown functions of θ whose forms are to be determined by the altered equations of motion. Let

$$T=\tfrac{1}{2}A_{11}\theta'^2+A_{12}\theta'\phi'+\tfrac{1}{2}A_{22}\phi'^2+A_{23}\phi'\psi'+\ldots\ldots\ldots\ldots (4),$$

where accents denote differential coefficients with regard to the time. Let also

$$T'=\tfrac{1}{2}A_{11}+A_{12}\phi_1+\tfrac{1}{2}A_{22}\phi_1^2+A_{23}\phi_1\psi_1+\ldots\ldots\ldots\ldots (5),$$

where the suffixes of ϕ, ψ, &c. here denote differentiations with regard to the new independent variable θ.

$$\therefore \frac{dT}{d\phi'}=\frac{dT'}{d\phi_1}\theta'; \qquad \frac{dT}{d\phi}=\frac{dT'}{d\phi}\theta'^2\ldots\ldots\ldots\ldots (6).$$

The equation of energy gives

$$T'\theta'^2 = U + C, \qquad \therefore\ \theta' = \left(\frac{U+C}{T'}\right)^{\frac{1}{2}} \quad \text{.....................} \quad (7).$$

The Lagrangian equation $\dfrac{d}{dt}\dfrac{dT}{d\phi'} - \dfrac{dT}{d\phi} = \dfrac{dU}{d\phi}$ becomes

$$\left(\frac{U+C}{T'}\right)^{\frac{1}{2}} \frac{d}{d\theta}\left\{\left(\frac{U+C}{T'}\right)^{\frac{1}{2}} \frac{dT'}{d\phi_1}\right\} = \frac{dT'}{d\phi}\,\frac{U+C}{T'} + \frac{dU}{d\phi},$$

where all the differential coefficients are partial except the $d/d\theta$.

Remembering that U is not a function of ϕ_1, this becomes

$$\frac{d}{d\theta}\frac{d}{d\phi_1}\{(U+C)\,T'\}^{\frac{1}{2}} = \frac{d}{d\phi}\{(U+C)\,T'\}^{\frac{1}{2}} \quad \text{...............} \quad (8).$$

If then we use $Q = \{(U+C)\,T'\}^{\frac{1}{2}}$ *as if it were the Lagrangian function and regard* θ *as the independent variable, we have the equations*

$$\frac{d}{d\theta}\frac{dQ}{d\phi_1} = \frac{dQ}{d\phi}, \qquad \frac{d}{d\theta}\frac{dQ}{d\psi_1} = \frac{dQ}{d\psi}, \ \&c. \quad \text{.....................} \quad (9),$$

from which the paths may be found.

This result follows easily from the theorem of Art. 524 by putting $d\tau = d\theta$, and we have here reproduced so much of that article as is required for our present purpose. If $d\tau = d\theta$, we have $Md\theta = dt$ and therefore by (7) of this note $M = \left(\dfrac{T'}{U+C}\right)^{\frac{1}{2}}$. Substituting in (2) the Lagrangian function becomes

$$L = 2\{(U+C)\,T'\}^{\frac{1}{2}}.$$

We notice that however the expressions for the vis viva and the work function may be different in different problems, yet *so long as the product* $(U+C)\,T'$ *remains unchanged, the paths are determined by the same relations between the coordinates* θ, ϕ, *&c.*

Since in the Lagrangian equations, the letters θ, ϕ, &c. represent arbitrary functions of the quantities or coordinates which determine the position of the system, it is evident that we have here taken as the independent variable any arbitrary function of the coordinates.

If some one coordinate, say ϕ, *is absent from the product* $(U+C)\,T'$ (though T contains the differential coefficients of ϕ), *we see that one solution of the equations of motion is*

$$\frac{dQ}{d\phi_1} = a, \qquad \therefore\ (U+C)^{\frac{1}{2}}\frac{dT'^{\frac{1}{2}}}{d\phi_1} = a \quad \text{.........................} \quad (10),$$

where a *is an arbitrary constant.* If C is arbitrary, the product Q cannot be independent of ϕ unless T' and U are separately independent of ϕ. But when C is given by the initial conditions this limitation is not necessary. If we substitute for $dT'/d\phi_1$ and T' the values given by (6) and (7) this integral becomes $dT/d\phi' = 2a$, which is the same as that obtained in Art. 521.

We may deduce this extension directly from the Lagrangian equations. Suppose

$$T = M\{\tfrac{1}{2}A_{11}\theta'^2 + \&c.\}, \qquad U + C = \frac{1}{M}f(\theta,\ \psi,\ \&c.),$$

where M is a function of θ, ϕ, &c. while A_{11}, &c. are not functions of ϕ. In this case the product $T(U+C)$ is not a function of ϕ. The Lagrangian equation

for ϕ gives

$$\frac{d}{dt}\frac{dT}{d\phi'} - \frac{dM}{d\phi}(\tfrac{1}{2}A_{11}\theta'^2 + \&c.) = -\frac{1}{M^2}\frac{dM}{d\phi}f(\theta, \psi, \&c.);$$

$$\therefore \frac{d}{dt}\frac{dT}{d\phi'} = \frac{dM}{d\phi}\frac{1}{M}(T - U - C) \quad\text{.........................(11).}$$

If then the initial circumstances are such that the equation of energy is $T = U + C$, we have $\dfrac{dT}{d\phi'} = \alpha$.

As a simple example, consider the case of a projectile moving under the action of gravity. We have $T = \frac{1}{2}(x'^2 + y'^2)$, $U = -gy$. Since the product of these is independent of x we choose some other coordinate as the independent variable. Writing $x_1 = dx/dy$ we have

$$Q = \{(1 + x_1^2)(y + C)\}^{\frac{1}{2}}; \qquad \therefore \frac{dQ}{dx_1} = \frac{x_1(y+C)^{\frac{1}{2}}}{\sqrt{(1 + x_1^2)}} = \alpha.$$

This by an easy integration leads to the parabola $(x - \beta)^2 = 4\alpha^2(y + C - \alpha^2)$.

The elimination of the time from the Lagrangian equations is given by Painlevé in his *Leçons sur l'intégration des équations différentielles de la Mécanique*, 1895. By an application of the principle of least action he obtains the function here called Q and writes the equations in the typical form $\dfrac{d}{dq_1}\dfrac{dQ}{dq'_n} = \dfrac{dQ}{dq_n}$. From these he deduces (page 239) that the Lagrangian equations may be written in the two forms

$$\frac{d}{dt}\frac{dT}{dq'} - \frac{dT}{dq} = \frac{dU}{dq}, \qquad \frac{d}{d\tau}\frac{dT'}{dq'} - \frac{dT'}{dq} = 0,$$

where $T' = T(U + C)$ and $d\tau = (U + C)\,dt$. This special result follows from that given at the beginning of this note by putting $1/M = U + C$. *Its importance lies in the fact that by this change the motion is made to depend on that of a system moving under no forces.*

The elimination of the time from Lagrange's equations is also given by Darboux in his *Leçons sur la théorie générale des surfaces*, Art. 571, 1889. He expresses his results in the same form as Painlevé.

We may obtain an extension of the theorem (2). In such problems as those discussed in Art. 255 the Lagrangian function takes the form

$$L = L_2 + L_1 + L_0 \quad\text{..................................(12),}$$

where L_n is a homogeneous function of θ', ϕ', &c. of the order n, the coefficients being functions of θ, ϕ, &c. but not of t. We then find as in Art. 512, Ex. 3, that the equation of energy becomes

$$L_2 - L_0 = C \quad\text{.......................................(13).}$$

Proceeding as in Art. 524, we change dt into $d\tau$ and write

$$L = \frac{L_2}{M} + L_1 + M(L_0 + C) \quad\text{.............................(14).}$$

We may now use this as the Lagrangian function.

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